Introduction to Dynamical Systems

Solutions Problem Set 1

Exercise 1. Carefully show that the map

$$\pi\colon \mathbb{R}^n \longrightarrow \mathbb{T}^n,$$

where $\mathbb{T}^n = S^1 \times S^1 \times \cdots \times S^1$ is the n-dimensional torus, given by

$$\pi(\mathbf{x}) = (e^{2\pi i x_1}, e^{2\pi i x_2}, \dots, e^{2\pi i x_n}), \quad \mathbf{x} = (x_1, \dots, x_n)$$

is a covering map.

Solution. We need only check that for each $y \in \mathbb{T}^n$ we may find a neighborhood $y \in U_y \subset \mathbb{T}^n$ such that

$$\pi^{-1}(U_y) = \bigsqcup_{\alpha \in A} V_\alpha,$$

where the V_α are open and pairwise disjoint, and $\pi|_{V_\alpha}$ is a homeomorphism between U_y and V_α. One can easily check that this is the case when letting $0 < \varepsilon < 1/2$ and $y = (e^{2\pi i \tilde{x}_1}, \dots e^{2\pi i \tilde{x}_n})$, and choosing

$$U_y = \prod_{j=1}^{n} \left\{ e^{2\pi i (x_j + \tilde{x}_j)} \colon -\varepsilon < x_j < \varepsilon \right\}, \quad V_{\mathbf{k}} = \prod_{j=1}^{n} (k_j - \varepsilon, k_j + \varepsilon),$$

where $\mathbf{k} = (k_1, \dots, k_n) \in A = \mathbb{Z}^n$. $\square$

Exercise 2. Show that Theorem 4.4 from lecture1.pdf is wrong in general if we only assume g to be Lebesgue integrable.

Solution. To show this, we only need to shift the rational numbers in $[0,1]$ by all the multiples of some irrational number $\alpha \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]$, and take f as the indicator function of the resulting set. Given such an α, define

$$Q_\alpha = (\mathbb{Q} + \alpha\mathbb{Z}) \cap [0,1], \quad f(x) = \mathbb{1}_{Q_\alpha}(x).$$

Then, for a given rational number $x \in [0,1]$ the left-hand side of the theorem reads

$$\frac{1}{N} \sum_{j=1}^{N} \mathbb{1}_{Q_\alpha}(\varphi_\alpha^j(x)) = \frac{1}{N} \sum_{j=1}^{N} 1 = 1,$$

whereas the right-hand side reads

$$\int_0^1 \mathbb{1}_{Q_\alpha}(x)\, \mathrm{d}x = |Q_\alpha| = 0,$$

given that Q_α is a countable set. $\square$

Exercise 3. Give a detailed proof of Theorem 4.7 in lecture1.pdf.

Solution. The proof follows the same structure and ideas as that of Theorem 4.4. Now $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, and the function $\varphi_{\boldsymbol{\alpha}}\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined via $\varphi(\mathbf{z}) = \mathbf{z} + \boldsymbol{\alpha}$, for $\mathbf{z} \in \mathbb{R}^n$. Trigonometric polynomials are now finite sums of terms of the form $f(\mathbf{z}) = e^{2\pi i \mathbf{n} \cdot \mathbf{z}}$, where

$\mathbf{n} \in \mathbb{Z}^n$ and $\cdot$ is the regular Euclidean product. The trivial case $\mathbf{n} = \mathbf{0}$ can be easily checked to work, and otherwise,

$$\sum_{j=0}^{N-1} f(\varphi_{\boldsymbol{\alpha}}^j(\mathbf{z})) = \sum_{j=0}^{N-1} e^{2\pi i \mathbf{n}\cdot(\mathbf{z}+j\boldsymbol{\alpha})} = e^{2\pi i \mathbf{n}\cdot\mathbf{z}} \frac{e^{2\pi i N \mathbf{n}\cdot\boldsymbol{\alpha}} - 1}{e^{2\pi i \mathbf{n}\cdot\boldsymbol{\alpha}} - 1},$$

which is bounded through the same considerations as in the proof of Theorem 4.4. Therefore,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(\varphi_{\boldsymbol{\alpha}}^j(\mathbf{z})) = 0 = \int_{[0,1]^n} f.$$

The Stone-Weierstrass theorem now grants density of trigonometric polynomials with respect to the L^∞–norm (uniform convergence) in $C([0,1]^n;\mathbb{R})$, and we only need to check the case of Riemann integrable functions. However, this part of the proof proceeds identically as that of Theorem 4.4. The only difference we need to take into account is that the Riemann integral is now defined in terms of partitions into rectangular subsets of the hypercube. □

Exercise 4. Show that the condition $\sum_{k=1}^n a_k \cdot j_k \notin \mathbb{Z}$ for $j_k \in \mathbb{Z}$ is satisfied for $n = 3$, $\alpha_1 = \sqrt{2}$, $\alpha_2 = \sqrt{3}$, $\alpha_3 = \sqrt{5}$.

Solution. We can easily check this by contradiction: assume that $a\sqrt{2} + b\sqrt{3} + c\sqrt{5} = k \in \mathbb{Z}$, where $a, b, c \in \mathbb{Z}$. Then, by rearranging and squaring both sides we reach

$$k^2 + 2a^2 - 2\sqrt{2}ak = 3b^2 + 5a^2 + 2bc\sqrt{15}.$$

Naming $r = k^2 + 2a^2 - 3b^2 - 5a^2 \in \mathbb{Z}$ and rearranging again, we find

$$r = 2\sqrt{2}ak + 2bc\sqrt{15}. \tag{1}$$

Squaring both sides again and renaming $p = r^2 - 8a^2k^2 - 60b^2c^2 \in \mathbb{Z}$,

$$p = 8akbc\sqrt{30}.$$

Assuming that the right-hand side is not zero, then we reach $\sqrt{30} \in \mathbb{Q}$. But this is false (as can be shown by an elementary contradiction argument).

On the other hand, for the right-hand side to be zero it means that either a, b, c or k is zero. If only one of a, b, c is nonzero, the result is trivially checked. Otherwise, (1) can be used to consider cases and check that we always reach a contradiction. □