

**INTRO TO DYNAMICAL SYSTEMS FALL 2024,
PROBLEM SET 6**

- (1) Let (X, m) be a probability space, i. e. a finite measure space with $m(X) = 1$. Letting σ be the underlying sigma algebra of measurable sets, introduce the set

$$\tilde{X} := \sigma / ,$$

i. e. the set of equivalence classes of measurable sets $A \in \sigma$, where we declare two measurable sets $A, B \in \sigma$ equivalent provided

$$m(A \Delta B) = 0.$$

- (i) Show that the function

$$d : \tilde{X} \times \tilde{X} \longrightarrow \mathbb{R}_+$$

given by (here $[A]$ is the equivalence class of measurable sets containing $A \in \sigma$)

$$d([A], [B]) = m(A \Delta B)$$

defines a metric on $\tilde{X}$.

- (ii) Show that $(\tilde{X}, d)$ is complete.

- (2) (Einsiedler-Ward) Let (X, m) be a probability space, and let $T : X \longrightarrow X$ a measure preserving map. Assume that for each pair of measurable $A, B \in \sigma$ there exists $N \in \mathbb{N}$ such that

$$m(A \cap T^{-n}B) = m(A) \cdot m(B)$$

for each $n \geq N$. Then show that $m(A) \in \{0, 1\}$ for each $A \in \sigma$.
Hint: use Baire category theorem and (1).

- (3) Let (X, m) be a finite measure space and $T : X \longrightarrow X$ measure preserving, bijective, with measurable inverse T^{-1} . Then show that for every $f \in L^1(X, dm)$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j(x)) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^{-j}(x))$$

for almost every $x \in X$.