INTRO TO DYNAMICAL SYSTEMS FALL 2024, PROBLEM SET 5

The following exercises are from Einsiedler-Ward.

(1) Let $\big(X, m\big)$ be a measure space with $m(X) = 1$, and let $T : X \longrightarrow X$ be measure preserving. Recalling the notation from last lecture, set $U_T f = f \circ T$. Then show that T is ergodic if and only if

$$\lim_{N\to\infty} N^{-1} \sum_{j=0}^{N-1} \langle U_T^j f, g\rangle = \langle f, 1\rangle \cdot \langle 1, g\rangle$$

for all $f, g \in L^2(X, m)$.

(2) Let $\big(X, m\big)$ a finite measure space, $T : X \longrightarrow X$ measure preserving, and let $f \in L^p\big(X, dm\big)$, $1 \le p < \infty$. Then show that there is $f^* \in L^p\big(X, dm\big)$, invariant under composition with T, and such that

$$\lim_{N\to\infty} N^{-1} \sum_{j=0}^{N-1} f\big(T^j x\big) = f^*(x)$$

with the convergence in the L^p-topology.

(3) Let X be a finite set and let $\sigma : X \to X$ a permutation. We call σ cyclic provided X is an orbit of σ.
Show that if $f : X \longrightarrow \mathbb{R}$ and σ is a cyclic permutation, then

$$\lim_{N\to\infty} N^{-1} \sum_{j=0}^{N-1} f(\sigma^j x) = \frac{1}{|X|} \cdot \sum_{x\in X} f(x).$$