INTRO TO DYNAMICAL SYSTEMS FALL 2024, PROBLEM SET 4

(1) Assume that ϕ, Φ are as in the structural stability theorem in lecture3.pdf. Also assume with

$$\Phi(x) = 2x + \hat{\psi}(x),\ \hat{\psi}(x+1) = \hat{\psi}(x),$$

that $\hat{\psi} \in C^1(\mathbb{R})$, and finally that $\hat{\psi}(0) = 0$, $\hat{\psi}'(0) \neq 0$. Then show that there is no open interval $(a, b) \subset \mathbb{R}$ on which the function u which solves

$$\Phi\big(u(x)\big) = u(2x) \tag{1}$$

is of class C^1. *Hint: first show that if u is C^1 on some open interval, then it is already C^1 everywhere. Then show that there is a dense set of points where the derivative u' vanishes.*

(2) (i) Let T be a contraction on the complete metric space $\big(X, d\big)$, with contraction factor $\lambda \in [0, 1)$, i. e. $d(Tx, Ty) \leq \lambda \cdot d(x, y)$. Then if x_* denotes the fixed point of T, show that for any $x_1 \in X$ there is a constant $C > 0$ such that

$$d\big(T^k(x_1), x_*\big) \leq C \cdot \lambda^k.$$

(ii) Let $Tg(x) = \Phi^{-1}\big(g(2x)\big)$ be as in lecture3.pdf, acting on the complete metric space X as in the lecture. By optimizing the choice of k in (i), show that there is a number $\alpha = \alpha(L) \in (0, 1)$ and a constant $C_* > 0$ such that

$$\big|u(x) - u(y)\big| \leq C_* \cdot \big|x - y\big|^\alpha$$

for every $x, y \in \mathbb{R}$ with $|x - y| \leq 1$. Here u solves the fixed point problem (1). Thus the conjugating function h is Holder continuous.

(3) Let X a finite measure space and $T : X \to X$ measure preserving. For measurable sets $A, B \subset X$, write

$$A \triangle B := (A \backslash B) \cup (B \backslash A).$$

Show that T is ergodic if and only if

$$m\big(A \triangle T^{-1}A\big) = 0$$

implies $m(A) = 0$ or $m(A) = m(X)$. *Hint:* $\bigcap_{n\geq 1} \bigcup_{j\geq n} T^{-j}(A)$.

(4) Let (X, m) be a finite measure space and $T : X \longrightarrow X$ a measure preserving map. Using the Von Neumann Mean Ergodic

Theorem, show that if $f \in L^1(X, dm)$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f\big(T^j(x)\big)$$

exists in $L^1(X, dm)$, and is a T-invariant function.