INTRO TO DYNAMICAL SYSTEMS FALL 2024, PROBLEM SET 10

(1) Let $A \in \mathrm{Mat}\big(n\times n, \mathbb{R}\big)$ a matrix whose eigenvalues $\lambda \in \mathbb{C}$ satisfy

$$\alpha < \mathrm{Re}\ \lambda < \beta$$

for two real numbers α, β. Show that there is an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\alpha \big\| x \big\|^2 \leq \langle Ax, x \rangle \leq \beta \big\| x \big\|^2,$$

where $\big\| x \big\|^2 = \langle x, x \rangle$.

(2) Consider the system of ODEs

$$\dot{\underline{y}} = \underline{f}(\underline{y})$$

where as usual $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{pmatrix}$. Assume that $\underline{0} \in \mathbb{R}^n$ is a fixed point, i. e. $f(\underline{0}) = \underline{0}$, and $f \in C^1\big(\mathbb{R}^n, \mathbb{R}^n\big)$. Finally, assume that

$$D\underline{f}(\underline{0}),$$

the Jacobian matrix, has only eigenvalues with negative real part. By using (1), show directly (i. e. without using Hartman-Grobman) that there exist $C > 0, c > 0$ and $\delta > 0$ such that for any initial condition

$$\underline{y}(0),\ \big\| \underline{y}(0) \big\| < \delta,$$

we have that

$$\big\| \underline{y}(t) \big\| \leq C \cdot e^{-ct} \cdot \big\| \underline{y}(0) \big\|.$$

(3) Give an example of a *non-diophantine irrational number* $\alpha \in \mathbb{R}$ in the sense of Lecture 10.