

PROOF OF HARTMAN-GROBMAN THEOREM, AND ITS APPLICATION TO SYSTEMS OF ODES

1. PROOF OF PROPOSITION 4.2

Relying on Lemma 4.3 from the previous lecture, we can now furnish a proof of Prop. 4.2, which in turn will lead to a proof of Hartman-Grobman:

Proof. (Prop. 4.2) We recall that we need to solve the equation

$$A\hat{h} - \hat{h} \circ \psi = \hat{\psi} - \hat{\phi} \circ h, \quad h(x) = x + \hat{h}(x).$$

Letting the expression on the left be $\mathcal{L}\hat{h}$, we have verified that $\mathcal{L}^{-1}$ acts boundedly on $C_b(\mathbb{R}^n, \mathbb{R}^n)$. We then reformulate the previous equation as

$$\hat{h} = \mathcal{L}^{-1}(\hat{\psi} - \hat{\phi} \circ h), \quad h(x) = x + \hat{h}(x).$$

which is a fixed point problem. To conclude that this admits a unique fixed point in $C_b(\mathbb{R}^n, \mathbb{R}^n)$, it suffices to show that the right hand expression is a contraction on this space. In fact, for

$$\hat{h}_{1,2} \in C_b(\mathbb{R}^n, \mathbb{R}^n),$$

setting $h_j(x) = x + \hat{h}_j(x)$, $j = 1, 2$, we have the estimate

$$\begin{aligned} & \sup_{x \in \mathbb{R}^n} \left\| \mathcal{L}^{-1}(\hat{\psi} - \hat{\phi} \circ h_1)(x) - \mathcal{L}^{-1}(\hat{\psi} - \hat{\phi} \circ h_2)(x) \right\| \\ &= \sup_{x \in \mathbb{R}^n} \left\| \mathcal{L}^{-1}(\hat{\phi} \circ h_1 - \hat{\phi} \circ h_2)(x) \right\| \\ &\leq \left\| \mathcal{L}^{-1} \right\| \cdot \varepsilon \cdot \sup_{x \in \mathbb{R}^n} \left\| (\hat{h}_1 - \hat{h}_2)(x) \right\|. \end{aligned}$$

Here ε is the Lipschitz constant associated with $\hat{\phi}$ and the norm $\| \cdot \|$:

$$\left\| \hat{\phi}(x) - \hat{\phi}(y) \right\| \leq \varepsilon \cdot \|x - y\|.$$

It follows that if $\varepsilon > 0$ is sufficiently small, so that $\left\| \mathcal{L}^{-1} \right\| \cdot \varepsilon < 1$, then the map

$$\hat{h} \longrightarrow T(\hat{h}) := \mathcal{L}^{-1}(\hat{\psi} - \hat{\phi} \circ h)$$

is a contraction, and hence admits a unique fixed point.

To see that the resulting map $h = \text{id} + \hat{h}$ is a homeomorphism of $\mathbb{R}^n$, we interchange the roles of ϕ, ψ , obtaining by an identical argument a map

$$g = \text{id}_{\mathbb{R}^n} + \hat{g}, \quad \hat{g} \in C_b(\mathbb{R}^n, \mathbb{R}^n)$$

with the property that

$$\psi \circ g = g \circ \phi.$$

But then

$$\psi \circ g \circ h = g \circ \phi \circ h = g \circ h \circ \psi.$$

Uniqueness of the map achieving the conjugation implies that

$$g \circ h = \text{id}_{\mathbb{R}^n}$$

and similarly

$$h \circ g = \text{id}_{\mathbb{R}^n}.$$

It follows that h is a homeomorphism.

Finally, we show that if $\phi(0) = \psi(0) = 0$, then $h(0) = 0$. In fact, since $\widehat{h}$ is obtained as fixed point of the contraction map T displayed above, it follows that

$$\widehat{h} = \lim_{j \rightarrow \infty} T^j(0).$$

From Lemma 4.3, we conclude inductively that $(T^j(0))(0) = 0$, and the claim follows from this. $\square$

2. PROOF OF HARTMAN-GROBMAN VIA REDUCTION TO PROPOSITION 4.2

The following lemma allows us to reduce the local information about the function ϕ to global information for a modified function.

Lemma 2.1. *Let $r > 0$. Then the function*

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

given by

$$f(x) = x, |x| \leq r$$

and

$$f(x) = r \cdot \frac{x}{|x|}, |x| \geq r$$

is Lipschitz continuous with Lipschitz constant 2 with respect to the standard Euclidean norm.

Proof. We need to show

$$|f(x) - f(y)| \leq 2|x - y|$$

We distinguish between the cases (i): $|x|, |y| \leq r$, (ii) $|x| > r, |y| \leq r$, and (iii): $|x|, |y| > r$.

The case (i) is clear, with Lipschitz constant 1.

In the case (ii), we have $|x| - r \leq |x| - |y| \leq |x - y|$, and so

$$\begin{aligned} |f(x) - f(y)| &= \left| r \cdot \frac{x}{|x|} - y \right| \\ &\leq |x| - r + \left| |x| \cdot \frac{x}{|x|} - y \right| \\ &\leq |x - y| + \left| |x| \cdot \frac{x}{|x|} - y \right| \\ &= 2|x - y| \end{aligned}$$

The remaining case (iii) is more of the same. $\square$

Finally we can provide

Proof. (Hartman-Grobman, Theorem 3.1 of Lecture 8) Let ϕ be as in the statement of Theorem 3.1 and write

$$\phi(x) = Ax + \widehat{\phi}(x), \quad A = D\phi(0).$$

Given $\varepsilon > 0$, there is a $r > 0$ such that we have for all $x \in B_r(0) \subset \mathbb{R}^n$

$$|D\widehat{\phi}(x)| < \varepsilon.$$

This then implies that if f is as in the preceding lemma with this constant r , then

$$\begin{aligned} |\widehat{\phi} \circ f(x_1) - \widehat{\phi} \circ f(x_2)| &\leq C \cdot \sup_{x \in B_r(0)} |D\widehat{\phi}(x)| \cdot |f(x_1) - f(x_2)| \\ &\leq 2C\varepsilon \cdot |x_1 - x_2|. \end{aligned}$$

We pick ε small enough such that setting

$$\tilde{\phi}(x) := Ax + \hat{\phi} \circ f(x), \tilde{\psi}(x) := Ax,$$

the hypothesis of Proposition 4.2 from Lecture 8 is satisfied. It follows that there is a homeomorphism

$$h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

with $h(0) = 0$ and such that

$$\tilde{\phi} \circ h = h \circ \tilde{\psi}.$$

Then if

$$V = h^{-1}(B_r(0)),$$

we have for $x \in V$

$$\tilde{\phi} \circ h(x) = \phi \circ h(x) = h \circ \tilde{\psi}(x).$$

Alternatively, setting $U = B_r(0)$, and then $h^{-1} : U \longrightarrow V$, we get

$$h^{-1} \circ \phi(x) = \tilde{\psi} \circ h^{-1}(x), \quad x \in U.$$

□

3. A VERSION OF HARTMAN-GROBMAN FOR SYSTEMS OF ODES

Consider an autonomous system of ordinary differential equations

$$(3.1) \quad \begin{aligned} \dot{y}_1 &= f_1(y_1, y_2, \dots, y_n), \\ \dot{y}_2 &= f_2(y_1, y_2, \dots, y_n), \\ &\dots \\ \dot{y}_n &= f_n(y_1, y_2, \dots, y_n). \end{aligned}$$

Here the functions $f_j \in C^1(\mathbb{R}^n, \mathbb{R})$. This implies in particular that for any initial condition

$$y(0) = (y_1(0), y_2(0), \dots, y_n(0)) \in \mathbb{R}^n,$$

there exist $t_0 > 0$ such that (3.1) admits a unique solution $y(t) \in C^1((-t_0, t_0), \mathbb{R}^n)$. Assume that $y_0 \in \mathbb{R}^n$ is a *fixed point* of (3.1). This means that

$$f_j(y_{0,1}, y_{0,2}, \dots, y_{0,n}) = 0, \quad j = 1, 2, \dots, n.$$

In particular the *constant function* $y_j(t) = y_{0,j}, j = 1, 2, \dots, n$, is a solution of (3.1).

Definition 3.1. The fixed point $y_0 \in \mathbb{R}^n$ is called a *hyperbolic fixed point* for the ODE system, provided the Jacobian matrix

$$Df(y_0) = (\partial_{y_j} f_k(y_0))_{\leq j, k \leq n}$$

has only eigenvalues with non-vanishing real part. Equivalently the matrix

$$e^{DF(y_0)}$$

is hyperbolic in the earlier sense.

The natural problem is to understand the *flow of the system* (3.1) for initial conditions $y_* \in \mathbb{R}^n$ close to y_0 , a hyperbolic fixed point.

The simplest situation arises of course when

$$\underline{f}(y) = A \cdot y,$$

where we set $\underline{f} = \begin{pmatrix} f_1 \\ f_2 \\ \dots \\ f_n \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix}$. Then the solution of (3.1) with initial condition $y(0) \in \mathbb{R}^n$ is given

by

$$y(t) = e^{tA}y(0).$$

We note the following lemma, whose proof is left as exercise.

Lemma 3.2. *If the matrix A is hyperbolic in the ODE system sense, so that e^A is hyperbolic in the sense of Lecture 8, then there is a splitting*

$$\mathbb{R}^n = E_+ \oplus E_-$$

as well as constants $c > 0, \theta_{\pm} \in (0, 1)$, such that we have $e^{tA}|_{E_{\pm}} : E_{\pm} \rightarrow E_{\pm}$, and further, we have

$$\begin{aligned} |e^{tA}y_0| &\leq c \cdot \theta_+^t \cdot |y_0|, \forall y_0 \in E_+, t \geq 0 \\ |e^{-tA}y_0| &\leq c \cdot \theta_-^t \cdot |y_0| \forall y_0 \in E_+, t \geq 0. \end{aligned}$$

The key issue now is to understand in what sense the preceding lemma, which gives the behavior of *linear systems of ODEs with hyperbolic fixed point* 0, reflects the behavior of a system (3.1) near a hyperbolic fixed point. In fact, another version of the Hartman-Grobman theorem addresses this issue. To formulate it, we denote the flow associated to (3.1) by $\phi(t, y_*)$, meaning the solution of this system with initial condition

$$y(0) = y_* \in \mathbb{R}^n$$

is given by

$$\phi(t, y_*)$$

at time t . Then the autonomous character of the system implies the following *flow property*:

$$\phi(t + s, y_*) = \phi(t, \phi(s, y_*)).$$

Assume now that $y_0 = 0$ is a hyperbolic fixed point for (3.1). Let $\phi(y_*) = \phi(1, y_*)$ be the time one map of the flow associated to (3.1). Note that we have

$$(3.2) \quad \underline{f}(y) = Df(0) \cdot y + F(y), \quad F(y) = o(|y|),$$

where we have $\lim_{|y| \rightarrow 0} |y|^{-1} \cdot o(|y|) = 0$. Then we can formulate

Lemma 3.3. *Letting $y_* \rightarrow \phi(y_*)$ be as before the time one map of the flow, we have that ϕ is a C^1 -map, and that $D\phi(0) = e^{Df(0)}$ is hyperbolic (i. e. all eigenvalues have absolute value $|\lambda| \neq 1$).*

Proof. (sketch) We formulate the solution of (3.1) with initial data $y(0) \in \mathbb{R}^n$ as solution of the fixed point problem

$$y(t) = \int_0^t e^{(t-s)Df(0)} \cdot F(y(s)) ds + e^{tDf(0)} \cdot y(0).$$

This admits a unique solution for small enough $y(0)$ by the Banach fixed point theorem. Moreover, we have that

$$\left| \int_0^t e^{(t-s)Df(0)} \cdot F(y(s)) ds \right| = o(|y(0)|) \forall t \in [0, 1].$$

It follows that

$$y(1) = \phi(1, y(0)) = e^{Df(0)} \cdot y(0) + o(|y(0)|),$$

which implies the lemma. □

Theorem 3.4. (Hartman-Grobman) *Let f , be as before, with fixed point 0 which is hyperbolic in the ODE-sense, i. e. such that $Df(0)$ has all eigenvalues with non-vanishing real part. Then there exist open neighborhoods U, V of 0 as well as a homeomorphism*

$$h : U \rightarrow V, \quad h(0) = 0,$$

such that we have

$$h \circ \phi(t, y(0)) = e^{tA} \cdot h(y(0)), \quad A = Df(0),$$

as long as $\phi(s, y(0)) \in U, y(0) \in U$ for all $0 \leq s \leq t$.

Proof. We will first modify f in order to ensure that the modified flow $\tilde{\phi}(t, y)$ is always globally defined and agrees with the original flow $\phi(t, y)$ provided the latter point is sufficiently close to 0. For this pick $r > 0$ and a C^1 -function $\underline{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with the property that $\underline{f}|_{B_r(0)} = \underline{f}|_{B_r(0)}$, and such that the flow $\tilde{\phi}(t, y)$ for the system

$$\dot{y} = \underline{f}(y)$$

is always globally defined. Moreover, we can arrange that $\underline{f}(y) = A \cdot y$ for $|y| \geq 2r$. By picking r suitably small, we can ensure that then

$$\tilde{\phi}(1, y) = e^A \cdot y + \hat{\phi}(y),$$

such that setting $\tilde{\psi} = e^A \cdot y$, the maps $\tilde{\phi}(1, y), \tilde{\psi}$ satisfy the assumptions of Proposition 4. 2 in Lecture 8 with A replaced by e^A . More generally, we have

$$\tilde{\phi}(t, y) = e^{tA} \cdot y + \hat{\phi}_t(y)$$

with $\hat{\phi}_t(y)$ bounded for t in a compact set and y arbitrary.

Exercise: check this carefully.

Proposition 4. 2 then allows us to conclude that there is a homeomorphism $\tilde{h} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with the properties that

$$(3.3) \quad \tilde{h} \circ \tilde{\phi}(1, \cdot) = e^A \circ \tilde{h}, \quad \tilde{h}(0) = 0.$$

Furthermore,

$$\tilde{h}(x) = x + \hat{h}(x)$$

with $\hat{h} \in C_b(\mathbb{R}^n, \mathbb{R}^n)$, and it is the unique map of this form with the preceding property (by the proof of Prop. 4.2).

Now we seek a homeomorphism $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with the property that

$$H \circ \tilde{\phi}(t, y) = e^{tA} \circ H.$$

Alternatively, we seek H such that

$$(3.4) \quad e^{-tA} \cdot H \circ \tilde{\phi}(t, y) = H(y)$$

for all $t \in \mathbb{R}$ and $y \in \mathbb{R}^n$.

To find such H , first introduce the following *continuous action of the additive group $\mathbb{R}$ on $C^0(\mathbb{R}^n; \mathbb{R}^n)$* , defined by

$$(3.5) \quad T_t g := e^{-tA} \cdot g \circ \tilde{\phi}(t, \cdot).$$

It is easily verified that this is indeed a group action in the sense that

$$T_t(T_s g) = T_{t+s}(g).$$

What we are looking for is a *fixed point for this action* namely

$$T_t(H) = H \quad \forall t \in \mathbb{R}.$$

Previously we have looked at *discrete group actions* (in case T was invertible) of the type $g \rightarrow g \circ T^j$, $j \in \mathbb{Z}$, which you can think of as an action of $\mathbb{Z}$ on the space of functions. The ergodic theorems have taught us that one way to arrive at an object *invariant under such a group action* is by *averaging*.

Now we have a continuous group actions, so averaging corresponds to computing an integral. So formally it would be natural to set

$$H = \int_{\mathbb{R}} T_s G ds$$

for some function G , since then *formally*

$$T_t H = \int_{\mathbb{R}} T_t(T_s G) ds = \int_{\mathbb{R}} T_{t+s} G ds = \int_{\mathbb{R}} T_s G ds = H.$$

The problem is that such an infinite integral won't converge in general (for non-zero G), and we would much rather replace this by a *finite integral*

$$H = \int_0^p T_s G ds,$$

say, for some $p > 0$. But then we get

$$T_t(H) = T_t\left(\int_0^p T_s G ds\right) = \int_t^{t+p} T_s G ds,$$

which differs in general from H , *except when $T_s G$ is periodic of period p* , i. e.

$$T_p G = G.$$

However, such a G is given to us for free, namely

$$G = \tilde{h}$$

with $\tilde{h}$ as in (3.3), which implies $p = 1$ and $T_1 \tilde{h} = \tilde{h}$. Hence we set

$$(3.6) \quad H = \int_0^1 T_s(\tilde{h}) ds,$$

This then indeed satisfies $T_t(H) = H$ for all $t \in \mathbb{R}$. Further, we have

$$\begin{aligned} H &= \int_0^1 e^{-sA} \cdot (id + \hat{h}) \circ (e^{sA} \cdot id + \hat{\phi}_s) ds \\ &= id + \hat{H}, \end{aligned}$$

where $\hat{H}$ is uniformly bounded. But using that $T_1 H = H$ which is also satisfied by $\tilde{h}$, the uniqueness of $\tilde{h}$ (solving $T_1 \tilde{h} = \tilde{h}$ with the fine structure just established) shows that

$$\tilde{h} = H.$$

This shows that $\tilde{h}$ already satisfies

$$T_t \tilde{h} = \tilde{h}$$

for all $t \in \mathbb{R}$, and $\tilde{h}$ is a homeomorphism. Explicitly

$$e^{-tA} \cdot \tilde{h} \circ \tilde{\phi}(t, y) = \tilde{h}(y)$$

But if $|\phi(s, y)|$ is small enough for all $0 \leq s \leq t$, then we have $\tilde{\phi}(t, y) = \phi(t, y)$, and so indeed we have that

$$e^{-tA} \cdot \tilde{h} \circ \phi(t, y) = \tilde{h}(y)$$

for such t, y . The Hartman-Grobman theorem then follows with $U = B_r(0)$ and $h = \tilde{h}$. □

4. A BASIC EXAMPLE: THE DAMPED CIRCULAR PENDULUM

Let a pendulum of mass m attached to a (massless) cord of length l be suspended from a point P in 3-space. We describe the motion in terms of the angle $\theta(t)$, measured with respect to the (vertically down) position of rest. The force driving the motion is the component of the gravitational force $(0, 0, -mg)$ tangential to the circle of motion, and a friction force proportional to the velocity. The magnitude of this force is $mg \sin \theta$, whence according to Newton's law of motion, we obtain

$$l\theta''(t) = -\sin \theta - \frac{kl}{m}\theta'$$

where we assume the friction force to be given by $-kl\theta'$, i. e. minus velocity times constant k . This is a second order scalar equation which we re-cast as a first order system by introducing $\omega = \theta'$, which results in

$$\begin{pmatrix} \theta' \\ \omega' \end{pmatrix} = \begin{pmatrix} \omega \\ -\frac{1}{l}\sin \theta - \frac{k}{m}\omega \end{pmatrix}$$

whence $\mathbf{f}(\theta, \omega)$ is given by the right hand side. Clearly the point $(\omega, \theta) = (0, 0)$ is an equilibrium, corresponding to the pendulum hanging down at rest, and we can compute

$$D\mathbf{f}(\mathbf{0}) = \begin{pmatrix} 0 & 1 \\ -\frac{1}{l} & -\frac{k}{m} \end{pmatrix}$$

with eigenvalues

$$\lambda_{1,2} = \frac{1}{2} \left[-\frac{k}{m} \pm \sqrt{\left(\frac{k}{m}\right)^2 - \frac{4}{l}} \right]$$

In particular, for nonzero friction $k > 0$, the eigenvalues have negative real parts, whence the equilibrium point $(0, 0)$ is a hyperbolic fixed point. Hartman-Grobman and one of your exercises immediately imply that sufficiently close excitations of this equilibrium converge back exponentially fast. This is of course not too surprising. Note that we can no longer draw this conclusion if $k = 0$. Indeed, in that case, energy conservation implies that the pendulum will swing back and forth forever. Note that the linearization has non-real eigenvalues provided $(\frac{k}{m})^2 < \frac{4}{l}$. This appears to suggest that the system will swing back and forth in this case (for small perturbations of the equilibrium), while we expect simple exponential approach to equilibrium otherwise.