THE STRUCTURE OF DIFFEOMORPHISMS AROUND HYPERBOLIC FIXED POINTS: HARTMAN-GROBMAN THEOREM

1. Introduction; hyperbolic isomorphisms

We follow Zehnder in this chapter. Recall the doubling map $T_2 : S^1 \longrightarrow S^1$, which is given by the simple expression $T_2(x) = 2x$ when working on the universal cover $\mathbb{R}$ of S^1. We have seen that suitably small perturbations of this map *can be conjugated back into the original map* by means of a homeomorphism, a fact we called *structural stability*. Thus if $\phi(x) = 2x + \widehat{\phi}(x)$ is sufficiently close to the doubling map in the sense that $\widehat{\phi}(x+1) = \widehat{\phi}(x)$ and $\widehat{\phi}$ satisfies a suitable smallness property (in terms of its Lipschitz constant), then there exists a homeomorphism $u : \mathbb{R} \longrightarrow \mathbb{R}$, $u(x+1) = u(x) + 1$, with the property that

$$u^{-1} \circ \phi \circ u = T_2.$$

It is natural to pose the question whether aspects of this phenomenon can be generalized to higher dimensions and more general maps

$$\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ \phi \in C^1\big(\mathbb{R}^n; \mathbb{R}^n\big),$$

which we assume to be diffeomorphisms.
It turns out that this question can be profitably investigated in a *perturbative setting around a fixed point*, which we may assume to be $x = 0 \in \mathbb{R}^n$. The idea then is to investigate the behavior of ϕ in a small neighborhood of the fixed point 0, where we may expect ϕ to be well approximated by its *linearization* $x \longrightarrow Ax$, $A = D\phi(0) \in \mathrm{Mat}(n \times n; \mathbb{R})$. We make the

Definition 1.1. *Let $A \in Mat(n \times n; \mathbb{R})$ an invertible matrix with real entries. Then we call A hyperbolic, provided all the (complex) eigenvalues λ of A satisfy*

$$\big|\lambda\big| \neq 1.$$

In other words, we can partition the eigenvalues into two sets where either $|\lambda| \in (0,1)$ or $|\lambda| \in (1,\infty)$.

What matters to us is a decomposition of $\mathbb{R}^n$ as a sum set of two components E_+, E_- on which A either grows or shrinks vectors asymptotically upon iteration:

Lemma 1.2. *Let A be a hyperbolic matrix. Then there exists a direct sum decomposition (here either one of the summands can be the trivial vector space)*

$$\mathbb{R}^n = E_+ \oplus E_-$$

and such that if we identify A with its multiplication action on $\mathbb{R}^n$ and label $A|_{E_\pm} = A_\pm$, there exists a positive constant $c \in \mathbb{R}$ as well as $\theta_\pm \in (0,1)$ such that (here $\big|\cdot\big|$ is a fixed norm on $\mathbb{R}^n$)

$$\big|A_+^n(x)\big| \leq c \cdot \theta_+^n \cdot \big|x\big|,\ x \in E_+,\ \big|A_-^{-n}(x)\big| \leq c \cdot \theta_-^n \cdot \big|x\big|,\ x \in E_-,\ n \geq 0.$$

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the eigenvalues with $|\lambda_j| \in (0,1)$, and $\lambda_{k+1}, \ldots, \lambda_n$ those with $|\lambda_j| \in (1,\infty)$. If $\lambda_i \in \mathbb{C} \backslash \mathbb{R}$, then its complex conjugate is also amongst the eigenvalues(since A has real entries), and of course of the same class. In this case, let

$$W_j \subset \mathbb{C}^n$$

be its generalized eigenspace, whence

$$\overline{W_j}$$

is the generalized eigenspace of $\overline{\lambda_j}$. If

$$\{e_{1j}, e_{2j}, \ldots, e_{i_j j}\} \subset \mathbb{C}^n$$

is a basis for W_j as $\mathbb{C}$-vectorspace, we consider the *real vectors*

$$f_{lj} = \frac{1}{2}(e_{lj} + \overline{e_{lj}}),\ g_{lj} = \frac{1}{2i}(e_{lj} - \overline{e_{lj}}),\ l = 1, 2, \dots, i_j.$$

If $\lambda_j \in \mathbb{R}$, then we can pick a real basis $\{h_{1j}, \dots, h_{i_j j}\}$ for W_j. If we declare $h_{lj} = 0 \in \mathbb{R}^n$ for non-real valued λ_j, and $f_{lj} = g_{lj} = 0$ for real valued λ_j, then we let

$$E_+ := \bigoplus_{j=1}^{k'} \mathrm{span}\{f_{lj},\, g_{lj},\, h_{lj}\}_{l=1}^{i_j},\ E_- := \bigoplus_{j=k+1}^{k''} \mathrm{span}\{f_{lj},\, g_{lj}, h_{lj}\}_{l=1}^{i_j};$$

Here the for the first space we include only those basis vectors corresponding to $|\lambda| < 1$ (in particular $k' \leq k$), while for the second we use those with $|\lambda| > 1$. We leave it as an exercise to verify that there is indeed a direct sum decomposition of $E_+ \oplus E_- = \mathbb{R}^n$, and that $E_\pm$ have the desired properties. For this recall the Jordan normal form. □

The fact that we have the constant c in the preceding lemma is a bit of a nuisance, since we would like the maps A_+, A_-^{-1} to act like contractions on the spaces E_+, E_-, respectively. For this the following lemma is useful:

Lemma 1.3. *Let $A \in Mat(n \times n; \mathbb{R})$ be a hyperbolic isomorphism of $\mathbb{R}^n$ and let $\theta_\pm$ be as in the preceding lemma. Then there exist constants*

$$\alpha \in (\theta_+, 1),\ \beta \in (\theta_-, 1)$$

and a norm $\|\cdot\|$ on $\mathbb{R}^n$ with the property that

$$\|A_+ x\| \leq \alpha \cdot \|x\|,\ \|A_-^{-1} x\| \leq \beta \cdot \|x\|.$$

Proof. This relies on an averaging trick: Choose $\alpha \in (\theta_+, 1), \beta \in (\theta_-, 1)$. Pick a large enough constant $N \geq 1$ such that letting c be the constant in the preceding lemma, we have

$$c \cdot \big(\frac{\theta_+}{\alpha}\big)^N \leq 1,\ c \cdot \big(\frac{\theta_-}{\beta}\big)^N \leq 1.$$

Then introduce the norm (with $\big|\cdot\big|$ the norm used in the previous lemma)

$$\|x\| := \sum_{j=0}^{N-1} \alpha^{-j} \cdot \big|A_+^j x\big|,\ x \in E_+,$$

and similarly

$$\|x\| := \sum_{j=0}^{N-1} \beta^{-j} \cdot \big|A_-^{-j} x\big|,\ x \in E_-.$$

We can then set

$$\|x\| := \max\{\|x_1\|, \|x_2\|\},$$

provided $x = x_1 + x_2$ with $x_1 \in E_+, x_2 \in E_-$, which defines $\|\cdot\|$ in general.
We claim that this is the desired norm. In fact, we have for $x \in E_+$

$$\|A_+ x\| = \sum_{j=0}^{N-1} \alpha^{-j} \cdot \big|A_+^{j+1} x\big| = \alpha \cdot \sum_{j=1}^{N} \alpha^{-j} \cdot \big|A_+^j x\big| = \alpha \cdot \big[\|x\| + \alpha^{-N}\big|A_+^N x\big| - \big|x\big|\big].$$

Here we have

$$\alpha^{-N}\big|A_+^N x\big| \leq \alpha^{-N} \cdot c\theta_+^N \cdot |x| \leq |x|$$

by choice of N. It follows that

$$\|A_+ x\| \leq \alpha \cdot \|x\|.$$

The argument for $x \in E_-$ is similar. □

2. The stable and unstable manifold of a hyperbolic fixed point

Definition 2.1. *Let $\phi \in C^1\big(\mathbb{R}^n;\mathbb{R}^n\big)$ be a diffeomorphism with fixed point $x_0 \in \mathbb{R}^n$. Then we say that x_0 is a hyperbolic fixed point, provided*

$$D\phi(x_0) \in Mat(n \times n; \mathbb{R})$$

is hyperbolic, whence gives rise to a hyperbolic isomorphism of $\mathbb{R}^n$.

It is straightforward to understand the *dynamics* of the discrete dynamical system $\big(\mathbb{R}^n, A\big)$ where $A \in \mathrm{Mat}(n \times n; \mathbb{R})$ is a fixed hyperbolic isomorphism which acts by multiplication. In fact, using the decomposition

$$\mathbb{R}^n = E_+ \oplus E_-,$$

we have that A maps each of the two sub spaces on the right into itself and we have that

$$E_+ = \{x \in \mathbb{R}^n,\ \lim_{n\to+\infty} A^n x = 0\},\ E_- = \{x \in \mathbb{R}^n,\ \lim_{n\to+\infty} A^{-n} x = 0\}.$$

We then call E_+ the *stable manifold* and E_- the *unstable manifold* associated to the (hyperbolic) fixed point $x_0 = 0$ for the map A. It is then natural to generalise these concepts to *diffeomorphisms*:

Definition 2.2. *Let $\phi \in C^1\big(\mathbb{R}^n;\mathbb{R}^n\big)$ be a diffeomorphism, and let $x_0 \in \mathbb{R}^n$ be a hyperbolic fixed point. Then we call*

$$W_+\big(\phi, x_0\big) := \{x \in \mathbb{R}^n,\ \phi^n(x) \longrightarrow x_0 \ \text{ as } n \to +\infty\}$$

the stable manifold of x_0. Similarly, we let

$$W_-\big(\phi, x_0\big) := \{x \in \mathbb{R}^n,\ \phi^{-n}(x) \longrightarrow x_0 \ \text{ as } n \to +\infty\}$$

the unstable manifold.

Our goal in the sequel will be to acquire a *local understanding* of these sets *near the fixed point x_0*. From now on we assume for simplicity that $x_0 = 0$. Let us suppose that there is a *homeomorphism* $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with the property that

$$h(0) = 0$$

and further such that

$$h \circ \phi \circ h^{-1}(x) = A \cdot x \tag{2.1}$$

where $A = D\phi(0)$. Then we have that

$$\phi^n(y) = h^{-1}\big(A^n(h(y))\big),$$

and hence we have that

$$y \in W_\pm\big(\phi, 0\big) \Longleftrightarrow h(y) \in E_\pm.$$

This means we can characterise the stable and unstable manifold at least as subsets of $\mathbb{R}^n$, albeit without any differentiability properties yet. Our goal is to give a partial solution for problem (2.1).

3. Statement of Hartman-Grobman

Finding a *global solution* of (2.1) is too ambitious since ϕ can be quite complicated in larger and larger sets, so we shall attempt to at least find h *locally*. This is accomplished in

Theorem 3.1. *Let $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a C^1-diffeomorphism, for which $x_0 = 0$ is a hyperbolic fixed point, i. e. $\phi(0) = 0$ and*

$$D\phi(0) \in Mat(n \times n; \mathbb{R})$$

is hyperbolic. Then there exist neighbourhoods U, V of 0 and a homeomorphism

$$h : U \longrightarrow V,$$

such that $h(0) = 0$ and

$$h \circ \phi(x) = A \circ h(x),$$

provided both x and $\phi(x)$ are in U. It follows that

$$h \circ \phi^j \circ h^{-1}(x) = A^j x,$$

provided $x \in V$ and $\phi^k(h^{-1}(x)) \in U$ for $k = 1, \ldots, j$.

The proof of this theorem, although in principle fairly elementary, requires a certain amount of preparations.

4. Perturbations of linear maps on Banach spaces

In order to prove the Hartman-Grobman theorem, we shall work near linear maps. The first order of the day is to have a precise quantitative understanding of such maps:

Lemma 4.1. *Let X be a Banach space and A a continuous linear isomorphism of X. Let*

$$\phi(x) = Ax + g(x),\, x \in X,$$

where g is a Lipschitz continuous map from X into itself with

$$\varepsilon \cdot \big\|A^{-1}\big\| < 1,$$

where ε is the Lipschitz constant for g. Then ϕ is a homeomorphism of X and ϕ^{-1} is Lipschitz continuous.,

Proof. (i): Injectivity of ϕ. Observe that if

$$Ax + g(x) = Ay + g(y),$$

then we obtain

$$x - y = -A^{-1}\big(g(x) - g(y)\big).$$

Then

$$\big|x - y\big| \leq \big\|A^{-1}\big\| \cdot \varepsilon \cdot \big|x - y\big|.$$

Since $\big\|A^{-1}\big\| \cdot \varepsilon < 1$, we conclude that $\big|x - y\big| = 0$, whence $x = y$.

(ii): Surjectivity. Given $y \in X$, we intend to solve the equation

$$y = Ax + g(x).$$

Then

$$x = A^{-1}y - A^{-1}g(x) =: T_y(x)$$

But the map $T_y(x)$ is a contraction due to the assumptions, and so admits a unique fixed point in X.

(iii): Lipschitz continuity of the inverse. Assume that

$$y = Ax + g(x),\, y' = Ax' + g(x').$$

Then

$$A^{-1}(y - y') = x - x' + A^{-1}(g(x) - g(x')).$$

By the triangle inequality we infer that

$$\begin{aligned}\big|A^{-1}(y - y')\big| &\geq \big|x - x'\big| - \big|A^{-1}(g(x) - g(x'))\big| \\ &\geq \big|x - x'\big| \cdot \big(1 - \big\|A^{-1}\big\| \cdot \varepsilon\big)\end{aligned}$$

It follows that

$$\big|x - x'\big| \leq \frac{\big\|A^{-1}\big\|}{1 - \big\|A^{-1}\big\| \cdot \varepsilon} \cdot \big|y - y'\big|.$$

□

The key for the proof of Hartman-Grobman is the following global perturbation result. We shall reduce to this result by a truncation trick:

Proposition 4.2. *Let $A \in Mat(n \times n; \mathbb{R})$ hyperbolic, and let $\phi, \psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by*

$$\phi(x) = Ax + \widehat{\phi}(x),\ \psi(x) = Ax + \widehat{\psi}(x),$$

where $\widehat{\phi}, \widehat{\psi}$ are bounded and Lipschitz continuous with sufficiently small Lipschitz constant ε. Then ϕ, ψ are homeomorphisms, and there exists a unique homeomorphism

$$h(x) = x + \widehat{h}(x),\ \big\|\widehat{h}\big\|_{L^\infty} < \infty,$$

conjugating ϕ into ψ:

$$\phi \circ h(x) = h \circ \psi(x),\ x \in \mathbb{R}^n.$$

If $\phi(0) = \psi(0) = 0$, we have $h(0) = 0$.

Let us see how to approach this: we compute

$$\begin{aligned} \phi \circ h(x) &= \phi \circ \big(Id + \widehat{h}\big)(x) = Ax + A\widehat{h}(x) + \widehat{\phi} \circ h(x), \\ h \circ \psi(x) &= Ax + \widehat{\psi}(x) + \widehat{h} \circ \psi(x). \end{aligned}$$

We conclude that we need to solve the following equation for $\widehat{h}$:

$$A\widehat{h} - \widehat{h} \circ \psi = \widehat{\psi} - \widehat{\phi} \circ h \tag{4.1}$$

Here the operator

$$\mathcal{L}\widehat{h} := A\widehat{h} - \widehat{h} \circ \psi$$

acts linearly. In fact, we shall let $\mathcal{L}$ act on the space of continuous bounded functions $C_b\big(\mathbb{R}^n, \mathbb{R}^n\big)$, equipped with the norm

$$\big|\widehat{h}\big| := \sup_{x \in \mathbb{R}^n} \big\|\widehat{h}(x)\big\|,$$

where we carefully use the norm

$$\big\|x\big\| = \max\{\big\|x_+\big\|,\ \big\|x_-\big\|\},$$

and $\big\|x_\pm\big\|$ are the norms constructed on $E_\pm$ as in Lemma 1.3. Then we require the following

Lemma 4.3. *Let A be a hyperbolic isomorphism. Then $\mathcal{L}$ is a continuous linear isomorphism of $C_b\big(\mathbb{R}^n, \mathbb{R}^n\big)$, and in particular its inverse*

$$\mathcal{L}^{-1}$$

is well-defined and bounded. If $\psi(0) = 0$ and $g(0) = 0$, we have that $\big(\mathcal{L}^{-1}g\big)(0) = 0$.

Proof. We show that the equation

$$\mathcal{L}v = g \tag{4.2}$$

admits a unique solution $v \in C_b\big(\mathbb{R}^n, \mathbb{R}^n\big)$ for given $g \in C_b\big(\mathbb{R}^n, \mathbb{R}^n\big)$. We do this by projecting this equation onto either component in the decomposition

$$\mathbb{R}^n = E_+ \oplus E_-.$$

In fact, decomposing for each $x \in \mathbb{R}^n$

$$v(x) = \sum_\pm v_\pm(x),\ v_\pm(x) \in E_\pm,$$

and using the fact that A maps $E_\pm$ into itself, we deduce that

$$A_\pm v_\pm - v_\pm \circ \psi = g_\pm, \tag{4.3}$$

where we denote $A|_{E_\pm} = A_\pm$. We reformulate these equations as follows[1]:

$$\begin{aligned} v_+ - A_+ v_+ \circ \psi^{-1} &= -g_+ \circ \psi^{-1}, \\ v_- - A_-^{-1} v_- \circ \psi &= A_-^{-1} g_-. \end{aligned} \tag{4.4}$$

[1]Recall that ψ is a homeomorphism.

The key point here is that the operators A_+, A_-^{-1} act as contractions on E_+, E_-, respectively, each equipped with $\|\cdot\|$. In fact, we have (for $\alpha \in [0,1)$)

$$\begin{aligned}\sup_{x\in\mathbb{R}^n}\big\|A_+v_+\circ\psi^{-1}(x) - A_+w_+\circ\psi^{-1}(x)\big\| &\leq \alpha\cdot\sup_{x\in\mathbb{R}^n}\big\|v_+\circ\psi^{-1}(x) - w_+\circ\psi^{-1}(x)\big\| \\ &= \alpha\cdot\sup_{x\in\mathbb{R}^n}\big\|v_+ - w_+\big\|.\end{aligned}$$

One proceeds similarly for $A_-^{-1}v_-\circ\psi$. The Banach fixed point theorem then implies the existence of unique $v_\pm \in C_b\big(\mathbb{R}^n,\, E_\pm\big)$ satisfying (4.4). This also gives the estimate

$$\sup_{x\in\mathbb{R}^n}\big\|\mathcal{L}^{-1}g(x)\big\| \leq C\cdot\sup_{x\in\mathbb{R}^n}\big\|g(x)\big\|$$

for a suitable constant C.
Finally, to see the last assertion of the lemma, setting for example $Tv_+ := A_+v_+\circ\psi^{-1}$ for $v_+ \in C_b\big(\mathbb{R}^n,\, E_+\big)$, we find that the solution of the first equation in (4.4) can be written as

$$v_+ = \sum_{j=0}^{\infty} T^j\big(-g_+\circ\psi^{-1}\big)$$

But if $\psi(0) = 0$ and $v(0) = 0$, then also $Tv(0) = 0$, and so inductively we infer that if $g(0) = 0$, $\psi(0) = 0$, we have

$$T^j\big(-g_+\circ\psi^{-1}\big)(0) = 0,\; j \geq 0.$$

This implies that then also $v_+(0) = 0$. One uses a similar argument for v_-. $\square$