THE EFFECT OF MEASURABILITY: MEASURE PRESERVING MAPS AND ERGODICITY

We now assume that X has the structure of a measure space. Thus there is a σ-algebra $\mathcal{A}$ of measurable subsets of X and a function

$$m : \mathcal{A} \longrightarrow [0, \infty]$$

which gives the measure of each of the sets $A \in \mathcal{A}$. We shall usually assume that $m(X) < \infty$, i. e. we are in the setting of a finite measure space.

1. Measure preserving maps

Now let $T : X \to X$ a map. We make

Definition 1.1. *We say that T is measurable, provided for each $A \in \mathcal{A}$, we have*

$$T^{-1}(A) \in \mathcal{A}.$$

We say that T is measure preserving, provided it is measurable and we have

$$m\big(T^{-1}(A)\big) = m(A)$$

for each $A \in \mathcal{A}$.

Is is easily seen that the rotation maps $z \to e^{2\pi i\alpha}z$, $\alpha \in \mathbb{R}$, are measure preserving, provided S^1 is equipped with the standard measure. We also mention

Example: *The doubling map $z \longrightarrow z^2$ from S^1 to itself is measure preserving.*

To see this, we first check that $m\big(T^{-1}(A)\big) = m(A)$ for each open interval $A = (a, b) \subset [0, 1)$, the latter interval being a fundamental domain for the standard projection map $\pi : \mathbb{R} \longrightarrow S^1$. Indeed, we have that

$$T^{-1}\big(A\big) = \big(\frac{a}{2}, \frac{b}{2}\big) \cup \big(\frac{a+1}{2}, \frac{b+1}{2}\big)$$

where we identify $T^{-1}\big(A\big)$ with a subset of the fundamental domain. But then

$$m\big(T^{-1}\big(A\big)\big) = \frac{b-a}{2} + \frac{b-a}{2} = b - a.$$

Next, if A is a general open set inside $[0, 1)$, we can write this as an at most countably infinite union of disjoint open intervals (in the relative topology), and so

$$m\big(T^{-1}\big(A\big)\big) = m(A)$$

follows from the countable additivity of the measure and the preceding case. Passing to complements, we see that T preserves measures of closed sets.
Finally, an arbitrary Lebesgue measurable subset $A \subset [0, 1)$ can be approximated arbitrarily well (in terms of the measure) from within and without by a closed set, respectively an open set. The fact that

$$m\big(T^{-1}\big(A\big)\big) = m(A)$$

Then follows from the preceding special cases.

We note that measure preservation can also be characterized in terms of the invariance of integrals as follows:

Lemma 1.2. *The map T is measure preserving if and only if*

$$\int_X f \circ T \, dm = \int_X f \, dm \tag{1.1}$$

for any $f \in L^1(X, dm)$.

Proof. If (1.1) holds, then also for $f = \chi_A$ for $A \in \mathcal{A}$ (which is an integrable function provided $m(X) < \infty$, which we assume here), and so

$$\int_X \chi_A \circ T \, dm = m\big(T^{-1}(A)\big) = \int_X \chi_A \, dm = m(A),$$

whence measure preservation follows.

On the other hand, assuming that T is measure preserving, we have that (1.1) holds for the characteristic functions of measurable sets, hence by linearity also for step functions, and from there for general integrable functions via approximation by step functions. □

We note that measure preservation is not enough to distinguish rational from irrational rotations of the circle, and a finer concept, namely *ergodicity*, will be introduced later to achieve this. Even so, there are some interesting facts concerning measure preserving maps in general, as discussed in the following section.

2. Poincaré recurrence

Let X be a *finite measure space*, and assume that $T : X \longrightarrow X$ is measure preserving. For a measurable set $A \in \mathcal{A}$, denote the set

$$A_0 = \{x \in A \,|\, T^j(x) \in A \text{ for infinitely many } j\}.$$

The following famous result due to Poincaré asserts that almost all points of A belong to A_0:

Theorem 2.1. *(Poincaré) Under the preceding assumptions, we have*

$$m(A) = m(A_0).$$

Proof. (Following Zehnder) We observe that

$$A_0^c = \bigcup_{n=1}^{\infty} C_n, \; C_n = \{x \in A, \, T^j(x) \notin A \, \forall j \geq n\}.$$

It suffices to show that $m(C_n) = 0$ for all $n \geq 1$. Note that

$$C_n = A \setminus \bigcup_{j \geq n} T^{-j}(A).$$

Clearly we have the inclusion

$$C_n \subset \bigcup_{j \geq 0} T^{-j}(A) \setminus \bigcup_{j \geq n} T^{-j}(A).$$

Since $\bigcup_{j \geq n} T^{-j}(A) \subset \bigcup_{j \geq 0} T^{-j}(A)$, we infer that

$$m(C_n) \leq m\big(\bigcup_{j \geq 0} T^{-j}(A)\big) - m\big(\bigcup_{j \geq n} T^{-j}(A)\big).$$

But since

$$T^{-n} \bigcup_{j \geq 0} T^{-j}(A) = \bigcup_{j \geq n} T^{-j}(A),$$

we deduce from the fact that T is measure preserving that

$$m\big(\bigcup_{j \geq 0} T^{-j}(A)\big) = m\big(\bigcup_{j \geq n} T^{-j}(A)\big).$$

We conclude that $m(C_n) = 0$ for each $n \geq 1$, as required. □

3. Ergodicity

We now introduce a further subclass of the measure preserving maps which in some sense gives the general analogue of the *irrational circle rotations*. This definition will play a fundamental role in the sequel:

Definition 3.1. *Let $T : X \to X$ be a measure preserving map. Then we say that T is ergodic, provided for every set $A \subset X$ satisfying*

$$T^{-1}\big(A\big) = A,$$

we have

$$m(A) = 0, \text{ or } m(X \backslash A) = 0.$$

There is an alternative classification in terms of invariant functions, as follows:

Lemma 3.2. *Let X be a finite measure space and $T : X \longrightarrow X$ measure preserving. Then T is ergodic if and only if the only function $f \in L^1(X, dm)$ which is left invariant by T in the sense of*

$$f\big(T(x)\big) = f(x)$$

for almost all $x \in X$ is the constant function (of course up to a set of measure zero).The same result holds of we replace L^1 by L^p, $1 \le p \le \infty$.

Proof. First assume that T is ergodic. If the conclusion is not true, then there is a real valued function f which is not constant almost everywhere and which is invariant under T. But then there is some number $c \in \mathbb{R}$ and such that

$$A := \big|\{x \in X,\ f(x) \ge c\}$$

satisfies

$$0 < m(A) < m(X).$$

But then the set A is invariant under T, since

$$f\big(T(x)\big) \ge c$$

if and only if $x \in T^{-1}(A)$, and $f\big(T(x)\big) = f(x)$. This contradicts ergodicity.

On the other hand, if there is no $f \in L^1(X, dm)$ left invariant under T, then there cannot be measurable $A \subset X$ with $T^{-1}(A) = A$, since else we could set $f = \chi_A$.

The last part of the lemma also easily follows since the characteristic function of a measurable set is automatically in these spaces, and they in turn are subspaces of L^1 due to the finiteness of the measure. □

We shall soon prove some fairly deep theorems which show that ergodicity is the right concept that allows us to generalize results such as Weyl's theorem. For now let us show that the two examples we encountered before, namely the irrational circle rotations, as well as the doubling map, are indeed ergodic.

Proposition 3.3. *Let $\alpha \in \mathbb{R} \backslash \mathbb{Q}$. Then the circle rotation $T : z \longrightarrow e^{2\pi i \alpha} \cdot z$ is ergodic.*

Proof. According to Lemma 3.2, it suffices to show that any $f \in L^2(X, dm)$ which is invariant under T is constant a. e., where $X = S^1$, equipped with the standard Lebesgue measure. So assume that f is invariant, and consider its Fourier series

$$\sum_{n \in \mathbb{Z}} e^{2\pi i n x} \cdot \widehat{f}(n),\ \widehat{f}(n) = \int_0^1 e^{-2\pi i n x} f(x)\, dx,$$

which converges in the L^2-norm towards f, where we identify S^1 with the fundamental domain $[0, 1)$. Then the function $f\big(T(x)\big)$ is given by

$$\sum_{n \in \mathbb{Z}} e^{2\pi i n (x + \alpha)} \cdot \widehat{f}(n) = \sum_{n \in \mathbb{Z}} e^{2\pi i n \alpha} \cdot e^{2\pi i n x} \cdot \widehat{f}(n).$$

But since this is the same function (except on a set of measure zero), we conclude that its Fourier coefficients have to agree with the ones of f, and hence

$$\widehat{f}(n) = e^{2\pi i n\alpha} \cdot \widehat{f}(n), \forall n \in \mathbb{Z}.$$

Since $e^{2\pi i n\alpha} \neq 1$ for $n \neq 0$, we conclude that

$$\widehat{f}(n) = 0$$

for all $n \neq 0$. This implies that the function f is constant(except on a set of measure zero). □

Similarly, we have the

Proposition 3.4. *The doubling map $T : z \longrightarrow z^2$ is ergodic.*

Proof. Again it suffices to show that there is no non-constant invariant function $f \in L^2(S^1)$. So assume $f \in L^2(S^1)$ is an invariant function, and consider the doubling map applied to it. Working with representatives on the fundamental domain $[0, 1)$ as before, and setting

$$g(x) = f(2x)$$

we compute

$$\begin{aligned}
\widehat{g}(2n) &= \int_0^1 e^{-2\pi i\cdot 2nx} \cdot f(2x)\, dx \\
&= \frac{1}{2}\int_0^2 e^{-2\pi i\cdot ny} \cdot f(y)\, dy \\
&= \frac{1}{2}\int_0^1 e^{-2\pi i\cdot ny} \cdot f(y)\, dy + \frac{1}{2}\int_1^2 e^{-2\pi i\cdot ny} \cdot f(y)\, dy \\
&= \int_0^1 e^{-2\pi i\cdot ny} \cdot f(y)\, dy \\
&= \widehat{f}(n),
\end{aligned}$$

where we have taken advantage of the periodicity of the 'lifting' of f to $\mathbb{R}$. If f is invariant under the doubling map, we find that

$$\widehat{f}(n) = \widehat{g}(2n) = \widehat{f}(2n).$$

Repeating this step with n replaced by $2n$, we conclude that

$$\widehat{f}(n) = \widehat{f}(2n) = \widehat{f}(4n) = \ldots = \widehat{f}(2^k n) = \ldots.$$

If $n \neq 0$, we have that $\left|2^k n\right| \longrightarrow +\infty$ as $k \to +\infty$, and so using the classical Riemann-Lebesgue Lemma, we conclude that

$$\widehat{f}(n) = 0,\ n \neq 0.$$

Again it follows that the function has to be constant (up to a set of measure zero). □

4. The Birkhoff ergodic theorem

We now try to see in what way the Weyl theorem can be generalized to a discrete dynamical system (X, T), with X a finite measure space, and *with an ergodic map T*. The following gives a very satisfying a natural answer:

Theorem 4.1. *(G. Birkhoff) Under the preceding assumptions, let $f \in L^1(X, dm)$. Then there exists a set $N \subset X$ of measure zero, and such that we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f\big(T^j(x)\big) = \frac{1}{m(X)} \cdot \int_X f\, dm,\ x \in X\backslash N.$$

This theorem is essentially the precise analogue of Weyl's theorem, except that the function f can be an arbitrary integrable one, but the only have the convergence for almost every orbit. To emphasize the difference, we formulate the special case

Corollary 4.2. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, and assume $f \in L^1(S^1)$. The if we set $T_\alpha(z) = e^{2\pi i\alpha} \cdot z$, $z \in S^1$, we have that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f\big(T_\alpha^j(z)\big) = \frac{1}{2\pi} \int_{S^1} f \, d\sigma$$

for almost every $z \in S^1$.

The proof of the corollary follows by combining Proposition 3.3 with the Birkhoff ergodic theorem.

Another corollary gives a kind of equidistribution result, but again of vastly more general character than in the circle setting:

Corollary 4.3. *Let X be as before and assume $A \subset X$ is measurable with $m(A) > 0$. Then if T is ergodic, we have that for almost each $x \in X$, the following limiting relation holds:*

$$\lim_{N\to\infty} \frac{1}{N} \cdot \big|\{j \in [0, \ldots, N-1],\ T^j(x) \in A\}\big| = \frac{m(A)}{m(X)}.$$

In particular, this gives a much more quantitative version of Poincaré recurrence in this setting.

The proof follows by applying the Birkhoff theorem to $f = \chi_A$. We note that the null set here may depend on A.