

STRUCTURAL STABILITY OF THE DOUBLING MAP

We follow again closely the treatment in Zehnder's book. Our goal will be to show that the orbit structure of the doubling map is unchanged under *small perturbations in the C^1 -topology*.

1. INTRODUCING STRUCTURAL STABILITY FOR CIRCLE MAPS

To begin with, we make

Definition 1.1. Let $\phi_0 : S^1 \rightarrow S^1$ a continuously differentiable map. Then we say that ϕ_0 is *structurally stable*, provided there is an $\varepsilon > 0$ such that the following holds: for every continuously differentiable mapping

$$\phi : S^1 \rightarrow S^1$$

satisfying the bound

$$\|\phi - \phi_0\|_{C^1(S^1)} < \varepsilon,$$

there exists a homeomorphism $h : S^1 \rightarrow S^1$, such that

$$(1.1) \quad \phi = h \circ \phi_0 \circ h^{-1}.$$

Remark 1.2. To define the norm $\|\cdot\|_{C^1(S^1)}$, we may use C^1 -liftings $\Phi_0, \Phi : \mathbb{R} \rightarrow \mathbb{R}$ of ϕ_0, ϕ , and require that

$$\|\Phi - \Phi_0\|_{C^1([0,1])} < \varepsilon.$$

Observe that since Φ_0 covers a circle map, we necessarily have

$$\Phi_0(x+1) = \Phi_0(x) + n \quad \forall x \in \mathbb{R},$$

for some $n \in \mathbb{Z}$. Then if Φ is close to Φ_0 in the preceding sense, meaning

$$\|\Phi - \Phi_0\|_{C^1([0,1])} = \|\Phi - \Phi_0\|_{L^\infty([0,1])} + \|(\Phi - \Phi_0)'\|_{L^\infty([0,1])} < \varepsilon,$$

and further Φ covers a circle map (namely ϕ), then necessarily also

$$\Phi(x+1) = \Phi(x) + n \quad \forall x \in \mathbb{R}.$$

Alternatively, we can write

$$(1.2) \quad \Phi(x) = \Phi_0(x) + \hat{\psi}(x),$$

where the map $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable, and

$$\hat{\psi}(x+1) = \hat{\psi}(x).$$

Remark 1.3. Note that if $\mathcal{O}^+(x)$ is a forward orbit of ϕ_0 , then

$$h(\mathcal{O}^+(x))$$

is the forward orbit of the point $h(x)$ under ϕ , since

$$h(\phi_0^j(x)) = (h \circ \phi_0 \circ h^{-1})^j(h(x)) = \phi^j(h(x)), \quad j \geq 0.$$

In particular, if ϕ_0 has a dense set of points x for which $\mathcal{O}^+(x)$ is dense, then the same holds for ϕ , since a homeomorphic image of a dense set is dense.

Remark 1.4. We note that h is only required to be a homeomorphism, which has one degree less of differentiability than the maps ϕ, ϕ_0 . In fact, in our setting below, this is the optimum one can hope for!

We now specialize $\phi_0(z) = z^2$, the doubling map. It admits the lifting $\Phi_0(x) = 2x$. A small perturbation of this map in the C^1 -norm admits a lifting with representation (1.2), i. e.

$$(1.3) \quad \Phi(x) = 2x + \hat{\psi}(x), \quad \hat{\psi}(x+1) = \hat{\psi}(x).$$

2. LIFTING (1.1) TO $\mathbb{R}$

Let us reflect what the equation (1.1) means in terms of Φ_0, Φ . If $h : S^1 \rightarrow S^1$ is a homeomorphism, we can lift this to a homeomorphism $u : \mathbb{R} \rightarrow \mathbb{R}$. Thus we have

$$(2.1) \quad h(e^{2\pi i x}) = e^{2\pi i u(x)}.$$

Since u is a homeomorphism on $\mathbb{R}$, it needs to be strictly monotonic, either increasing or decreasing. We shall consider

$$u(x) = x + \widehat{u}(x), \quad \widehat{u}(x+1) = \widehat{u}(x).$$

Then if u is strictly increasing, it is a homeomorphism of $\mathbb{R}$, since $\widehat{u}$ is a bounded function (being periodic and continuous), and hence

$$\lim_{x \rightarrow +\infty} u(x) = +\infty, \quad \lim_{x \rightarrow -\infty} u(x) = -\infty.$$

This implies via the intermediate value theorem that u is surjective. Since it is also injective by assumption, it is a homeomorphism of $\mathbb{R}$. Moreover, we can then define h via (2.1). The inverse of h is then given via

$$h^{-1}(e^{2\pi i x}) = e^{2\pi i u^{-1}(x)}.$$

Furthermore, we also have

$$u^{-1}(x+1) = u^{-1}(x) + 1,$$

for

$$u(u^{-1}(x) + 1) = u(u^{-1}(x)) + 1 = x + 1.$$

We now write (1.1) in the form

$$\phi \circ h = h \circ \phi_0,$$

At the level of the liftings, the preceding equation will be satisfied, provided

$$(2.2) \quad \Phi(u(x)) = u(\Phi_0(x)).$$

Theorem 2.1. *Assume that $\Phi(x) = 2x + \widehat{\psi}(x)$ as in (1.3), and further that $\widehat{\psi}$ is Lipschitz with constant $L \in [0, 1]$:*

$$|\widehat{\psi}(x) - \widehat{\psi}(y)| \leq L|x - y|.$$

Then there is a unique strictly increasing homeomorphism $u : \mathbb{R} \rightarrow \mathbb{R}$ with $u(x+1) = u(x) + 1$ and such that (2.2) is satisfied.

Proof. (Zehnder) To begin with, note that the assumption on $\widehat{\psi}$ and (1.3) imply that Φ is a homeomorphism on $\mathbb{R}$. In fact, for $x \geq y$ we observe the important inequalities

$$\begin{aligned} \Phi(x) - \Phi(y) &= 2(x - y) + \widehat{\psi}(x) - \widehat{\psi}(y) \geq 2(x - y) - L(x - y) = (2 - L)(x - y) =: r_1(x - y) \\ \Phi(x) - \Phi(y) &= 2(x - y) + \widehat{\psi}(x) + \widehat{\psi}(y) \leq 2(x - y) + L(x - y) = (2 + L)(x - y) =: r_2(x - y), \end{aligned}$$

where $r_1 > 1, r_2 < 3$. Fixing $y \in \mathbb{R}$ and letting $x \rightarrow \infty$ implies $\lim_{x \rightarrow +\infty} \Phi(x) = +\infty$, and similarly fixing x and letting $y \rightarrow -\infty$, we infer that $\lim_{y \rightarrow -\infty} \Phi(y) = -\infty$. By the intermediate value theorem, Φ is surjective, and by the preceding Φ is also injective, with a continuous inverse.

Furthermore, writing $x = \Phi^{-1}(x')$, $y = \Phi^{-1}(y')$, the preceding also gives

$$r_1(\Phi^{-1}(x') - \Phi^{-1}(y')) \leq x' - y' \leq r_2(\Phi^{-1}(x') - \Phi^{-1}(y')), \quad x' \geq y',$$

which is equivalent to

$$r_2^{-1}(x' - y') \leq \Phi^{-1}(x') - \Phi^{-1}(y') \leq r_1^{-1}(x' - y').$$

Applying absolute values, this results in

$$|\Phi^{-1}(x') - \Phi^{-1}(y')| \leq r_1^{-1}|x' - y'|.$$

In other words, the map $\Phi^{-1} : \mathbb{R} \rightarrow \mathbb{R}$ is a contraction.

We now apply Φ^{-1} to (2.2), which we reformulate as

$$u(x) = \Phi^{-1}(u(\Phi_0(x))) = \Phi^{-1}(u(2x)).$$

This is a kind of *fixed point equation*, which we solve in a suitable complete metric space via the Banach fixed point theorem. For this let

$$X := \{u \in C^0(\mathbb{R}; \mathbb{R}), u(x+1) = u(x) + 1, u \text{ increasing}\}.$$

We equip this with the metric (for $u, v \in X$)

$$d(u, v) := \sup_{x \in \mathbb{R}} |u(x) - v(x)| = \max_{0 \leq x \leq 1} |u(x) - v(x)|.$$

Here we importantly use that the difference $u(x) - v(x)$ is one periodic, and it suffices to restrict the argument x to the interval $[0, 1]$. It is easily checked that (X, d) is a complete metric space¹.

We now check that, if we denote

$$Tu(x) := \Phi^{-1}(u(2x)),$$

then

- T maps X into itself.
- T is a contraction on X .

For the first point, we observe that

$$Tu(x+1) = \Phi^{-1}(u(2x+2)) = \Phi^{-1}(u(2x) + 2)$$

On the other hand, we have (see (1.3))

$$\Phi(x+1) = 2x+2 + \widehat{\psi}(x) = \Phi(x) + 2 \longrightarrow \Phi^{-1}(y+2) = \Phi^{-1}(y) + 1.$$

It follows that

$$Tu(x+1) = \Phi^{-1}(u(2x)) + 1 = Tu(x) + 1.$$

Further, Tu is a continuous function, and as composition of increasing functions, it is increasing as well. This shows that $T : X \longrightarrow X$.

We next show that T is a contraction. But this is easy, since

$$\begin{aligned} d(Tu, Tv) &= \sup_{x \in \mathbb{R}} |\Phi^{-1}(u(2x)) - \Phi^{-1}(v(2x))| \\ &\leq r_1^{-1} \cdot \sup_{x \in \mathbb{R}} |u(2x) - v(2x)| \\ &= r_1^{-1} \cdot d(u, v). \end{aligned}$$

Since $r_1 > 1$, we conclude that T is indeed a contraction on X .

We are now almost done with the proof of the theorem. It only remains to show that u is *strictly increasing*, so that it is indeed a homeomorphism on S^1 . Note that since $u(x+1) = u(x) + 1$ and hence

$$u(x+n) = u(x) + n, n \in \mathbb{Z},$$

the surjectivity of the map u follows directly.

To get the injectivity, we again use a 'enlargement of scale' type argument, like we did in the proof of transitivity of the doubling map. Assume that there are $x_1 < x_2$ in $\mathbb{R}$ such that

$$u(x_1) = u(x_2).$$

Then since u increases, we find $u(x) = u(x_1) = u(x_2)$ for every $x \in [x_1, x_2]$. Pick an interval

$$I = [\frac{k}{2^n}, \frac{k+1}{2^n}] \subset [x_1, x_2].$$

¹Note that increasing does not mean strictly increasing!

Then

$$u\left(\frac{k}{2^n}\right) = \Phi^{-1}\left(u\left(\frac{k}{2^{n-1}}\right)\right) = u\left(\frac{k+1}{2^n}\right) = \Phi^{-1}\left(u\left(\frac{k+1}{2^{n-1}}\right)\right).$$

Since Φ^{-1} is injective, this implies that

$$u\left(\frac{k}{2^{n-1}}\right) = u\left(\frac{k+1}{2^{n-1}}\right).$$

But then we can repeat the preceding step to reduce $n-1$ to $n-2$ and so forth, until we have the relation

$$u(k) = u(k+1).$$

This, however, contradicts the definition of X , which stipulates that

$$u(k+1) = u(k) + 1.$$

This shows that u is indeed strictly increasing, and hence a homeomorphism of $\mathbb{R}$. $\square$

A natural question to ask is whether the homeomorphism h may be chosen of regularity C^1 if, say, Φ is of regularity C^1 . In fact, *this is not the case generically*:

Proposition 2.2. *Let Φ be as in (1.2) with $\widehat{\psi} \in C^1(\mathbb{R})$. Further assume that $\widehat{\psi}(0) = 0$ and $\widehat{\psi}'(0) \neq 0$. Then there is no diffeomorphism $h \in C^1(S^1; S^1)$ such that (1.1) holds.*

Proof. We again work at the level of the lifting. Assume $\Phi(x) = 2x + \widehat{\psi}(x)$, and assume that u satisfies (2.2). Then (evaluating at $x = 0$)

$$2u(0) + \widehat{\psi}(u(0)) = u(0) \longrightarrow u(0) + \widehat{\psi}(u(0)) = 0,$$

and

$$\widehat{\psi}(u(0)) = \widehat{\psi}(u(0)) - \widehat{\psi}(0) \in [-L|u(0)|, L|u(0)|].$$

This implies that $u(0) = 0$, since $L \in [0, 1)$.

Next differentiate (2.2). We conclude

$$\Phi'(u(0)) \cdot u'(0) = \Phi'(0) \cdot u'(0) = 2u'(0),$$

and so

$$(2 + \widehat{\psi}'(0)) \cdot u'(0) = 2u'(0)$$

Since by assumption $2 + \widehat{\psi}'(0) \neq 2$, this implies that

$$u'(0) = 0$$

which is not allowed since h is a diffeomorphism. $\square$