TRANSITIVITY IN DYNAMICAL SYSTEMS

1. A definition

In last lecture we studied the example of circle rotations $\phi : S^1 \longrightarrow S^1$, which have the property that they are either periodic (for $\alpha \in \mathbb{Q}$), or have orbits which are dense (for $\alpha \in \mathbb{R}\backslash\mathbb{Q}$). We shall soon study a more complex map where orbits can be both periodic (with arbitrary given period) as well as dense (all for the same map!).
This shall reveal that the existence of a dense orbit is often a delicate and non-obvious feature. The following definition highlights the presence of a dense orbit for a map:

Definition 1.1. *Assume $\big(X, d\big)$ is a metric space, and let $T : X \to X$ a continuous map. Then the (discrete) dynamical system $\big(X, T\big)$ is said to be transitive, provided there exists an orbit*

$$\mathcal{O}^+(x)$$

for some $x \in X$ which is dense, i. e. such that

$$\overline{\mathcal{O}^+(x)} = X.$$

The dynamical system $\big(X, T\big)$ is called minimal, provided every orbit of T is dense.

Example 1: Every circle rotation with $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is minimal. Every circle rotation with rational α is non-transitive.

Example 2: If $\alpha \in \mathbb{R}^n$ is such that $\sum_{k=1}^n \alpha_k \cdot j_k \notin \mathbb{Z}$ for any $\mathbf{j} = (j_k) \in \mathbb{Z}\backslash\{0\}$, then the map

$$T(\mathbf{z}) := \big(e^{2\pi i \alpha_k} z_k\big)_{k=1}^n : \mathbb{T}^n \longrightarrow \mathbb{T}^n$$

is transitive, and in effect minimal.

Example 3: We shall soon see that the 'doubling map' $z \longrightarrow z^2$ from S^1 to itself is *transitive but not minimal.*

Ona may also ask whether there are transitive maps on a non-compact domain, such as the plane. This is indeed the case as an example by A. Besicovitch (1937) shows, but the construction is somewhat involved (but excellent for a semester project!).

2. Birkhoff's transitivity theorem

Assume that $\big(X, d\big)$ is a metric space, and that $T : X \longrightarrow X$ is transitive. Thus there is a trajectory $\mathcal{O}^+(x)$ which is dense in X, and hence 'hits' every open set $U \subset X$. It may then be intuited that applying iterates of T to an arbitrary open set U should result in sets $T^n(U)$ which intersect any other open set $V \subset X$. It turns out that this is indeed the case under a certain technical restriction on $\big(X, d\big)$, and that moreover the opposite implication also holds (this is much more interesting than the first implication), under a different and more subtle technical restriction on $\big(X, d\big)$.

Theorem 2.1. *Assume $\big(X, d\big)$ is a metric space and $T : X \longrightarrow X$ is a continuous map.*
(i) If (X, d) does not have an isolated point, and if T is transitive, then for every pair of non-empty open sets U, V, there exists $n \geq 0$ such that

$$T^n\big(U\big) \cap V \neq \emptyset.$$

(ii) If $\big(X, d\big)$ is complete and has a countable basis of open sets, then if for every pair of nonempty open sets U, V there is $n \geq 0$ such that

$$T^n\big(U\big) \cap V \neq \emptyset,$$

then T acts transitively. In fact, the set of points with dense orbit is dense.

Proof. (i) Assume that $p \in X$ is such that its orbit is dense, $\overline{\mathcal{O}^+(p)} = X$. Let U, V be nonempty open sets. Then there is a $k_1 \geq 0$, which we may pick as small as possible, such that

$$T^{k_1}(p) \in V,$$

and since by assumption $V \neq \{T^{k_1}(p)\}$, and the set $\{T^{k_1}(p)\}$ is closed, we have that

$$V \backslash \{T^{k_1}(p)\}$$

is open and nonempty. Hence there is $k_2 > k_1$ minimal with the property that

$$T^{k_2}(p) \in V \backslash \{T^{k_1}(p)\}.$$

Again by assumption we have $V \backslash \{T^{k_1}(p)\} \neq \{T^{k_2}(p)\}$, and so the set

$$V \backslash \{T^{k_1}(p), T^{k_2}(p)\}$$

is nonempty and open. Continuing inductively we find a sequence $k_1 < k_2 < \ldots < k_n$ such that

$$T^{k_n}(p) \in V,\, n \geq 1.$$

Further, by assumption there is $l \geq 0$ such that $T^l(p) \in U$. Then pick n_* large enough such that $k_{n_*} \geq l$. Then we have

$$T^{k_{n_*} - l}\big(U\big) \cap V \neq \emptyset.$$

(ii) Let U, V be non empty open sets. By assumption, there is $n \geq 0$ such that

$$T^n(U) \cap V \neq \emptyset.$$

This in turn is equivalent to the fact that denoting the pre-images of V by

$$T^{-j}(V) := \{x \in X \,|\, T^j(x) \in V\},$$

then

$$\cup_{j \geq 0} T^{-j}(V) \cap U \neq \emptyset.$$

Since U is an arbitrary nonempty open set, this means that

$$\cup_{j \geq 0} T^{-j}(V) =: \mathcal{O}^-(V)$$

is dense in X, since we can set $U = B_\delta(x)$ for arbitrary $x \in X$ and $\delta > 0$.

Now let $\{V_j\}_{j \geq 1}$ be a countable basis of open (nonempty) sets. This implies in particular that for every open nonempty set $V \subset X$ there is some $V_j \subset V$. As observed before each $\mathcal{O}^-(V_j)$ is open and dense. Then we invoke the famous

Theorem 2.2. *(Baire category theorem) Let (X, d) be a complete metric space, and let A_j, $j = 1, 2, \ldots$ a countable collection of closed sets such that*

$$\bigcup_j A_j$$

has non-empty interior. Then at least one of the A_j has non-empty interior.

We actually need the 'complemented version' of this. Assume that B_j, $j \geq 1$ is a countable collection of open and dense sets. Then $\{B_j^c\}_{j \geq 1}$ is a collection of closed sets with empty interior. According to the preceding theorem, this implies that

$$\bigcup_j B_j^c$$

has empty interior. Taking complements, we infer that

$$\bigcap_j B_j$$

is dense.

Applying this observation with $B_j = \mathcal{O}^-(V_j)$, we find that

$$\bigcap_j \mathcal{O}^-(V_j)$$

is dense. But if

$$x \in \bigcap_j \mathcal{O}^-(V_j),$$

then its forward orbit is dense in X. In fact, given $U \subset X$ open and non empty, there is

$$V_l \subset U$$

for some l, and since

$$x \in \mathcal{O}^-(V_l),$$

there is some $k \geq 0$ with the property that

$$T^k(x) \in V_l.$$

□

Remark 2.3. The conditions for (X, d) in (ii) are for example satisfied when X is a compact metric space. Then completeness is clear, and for the countable base, it suffices to consider for each $k \geq 1$ a covering of X by the open balls

$$B_{\frac{1}{k}}(x),\ x \in X.$$

By compactness, we can select a finite collection

$$B_{\frac{1}{k}}(x_{kl}),\ l = 1, 2, \ldots, n_k$$

which still covers X, and the collection

$$\{B_{\frac{1}{k}}(x_{kl}),\ k = 1, 2,\ \ldots,\ l = 1,\ 2,\ \ldots, n_k\}$$

is countable and forms a basis.

For completeness' sake, we also provide a proof of the Baire categorical theorem.

Proof. (Baire) Arguing buy contradiction, we assume that none of the A_j has non empty interior. Pick a ball

$$B_{\delta_0}(p_0) \subset \bigcup_{j \geq 1} A_j,\ \delta_0 > 0.$$

Then there is some $p_1 \in B_{\delta_0}(p_0) \backslash A_1$ since A_1 has empty interior and in particular $B_{\delta_0}(p_0) \not\subseteq A_1$. By closedness of A_1, there is a $\delta_1 < \frac{\delta_0}{2}$ and such that

$$\overline{B_{\delta_1}(p_1)} \subset B_{\delta_0}(p_0) \backslash A_1.$$

Since $B_{\delta_1}(p_1) \not\subseteq A_2$, we can find $p_2, 0 < \delta_2 < \frac{\delta_1}{2}$ such that

$$\overline{B_{\delta_2}(p_2)} \subset B_{\delta_1}(p_1) \backslash A_2.$$

Proceeding inductively, we construct a sequence $p_k, 0 < \delta_k$ such that

$$\overline{B_{\delta_k}(p_k)} \subset B_{\delta_{k-1}}(p_{k-1}) \backslash A_k,\ \delta_k < \frac{\delta_{k-1}}{2}.$$

This implies moreover that

$$\overline{B_{\delta_k}(p_k)} \subset \big(\bigcup_{1 \leq j \leq k} A_j \big)^c,$$

and that the p_k form a Cauchy sequence. By completeness this sequence converges to a limit p_*, and we have

$$p_* \in \bigcap_{k \geq 1} \overline{B_{\delta_k}(p_k)} \subset \big(\bigcup_{1 \leq j < \infty} A_j \big)^c.$$

But at the same time we have

$$p_* \in B_{\delta_0}(p_0) \subset \bigcup_{j \geq 1} A_j,$$

a contradiction. □

3. An interesting example of a transitive dynamical system; the doubling map of the circle

We consider now a map for which the orbits can be both periodic as well as dense. In fact, there is a dense set of points giving rise to either type of orbit. This map is the innocuously looking *doubling map* $\phi : S^1 \longrightarrow S^1$ given by

$$\phi(z) = z^2.$$

As usual we can lift this to a map $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$\Phi(x) = 2x.$$

However, for later purposes, namely the remarkable identification with the *shift map*, it is better to use a different lifting, which however agrees with Φ modulo $\mathbb{Z}$, namely

$$\widehat{\Phi}(x) = \begin{cases} 2x,\ x \in [0, \frac{1}{2}) \\ 2x - 1,\ x \in [\frac{1}{2}, 1) \end{cases}$$

and continued periodically. The nice thing about $\widehat{\Phi}$ is that it maps the fundamental domain $[0, 1)$ into itself.

The action of $\widehat{\Phi}$ is particularly neatly expressed in terms of *binary expansions* of points in $[0, 1)$. In fact, every $x \in [0, 1)$ can be written in the form

$$x = \frac{x_1}{2} + \frac{x_2}{4} + \ldots + \frac{x_n}{2^n} + \ldots,$$

where we make the assumption that there are always infinitely many non-zero digits, to make the representation unique. Then note that

$$\widehat{\Phi}(x) = \frac{x_2}{2} + \ldots + \frac{x_n}{2^{n-1}} + \ldots.$$

This means that if we identify $[0, 1)$ with the set

$$\mathcal{F}_0 := \{(x_n)_{n=0}^{\infty},\ x_n \in \{0, 1\}, \text{ infinitely many members non-zero}\},$$

then the map $\widehat{\Phi}$ becomes the *shift map* $\sigma : \mathcal{F}_0 \longrightarrow \mathcal{F}_0$ defined by

$$\sigma\big(x_1, x_2, \ldots\big) = \big(x_2, x_3, \ldots\big).$$

This identification shall be useful later on when establishing the existence of lots of dense orbits via the Birkhoff transitivity theorem.

Here is the first remarkable result about the doubling map:

Theorem 3.1. *(i) The set of points $z \in S^1$ giving rise to a periodic orbit for ϕ is countable and dense.*
(ii) The set of points $z \in S^1$ with $\overline{\mathcal{O}^+(z)} = S^1$ is dense in S^1.

Proof. (i) The orbit of z is periodic precisely if there is $n \geq 1$ such that

$$\phi^n(z) = z \Longleftrightarrow z^{2^n - 1} = 1.$$

Thus z is gives rise to an orbit of period at most n precisely if z is a $2^n - 1$-th root of unity. These are of course equally spaced on S^1 for fixed n, and their union over all $n \geq 1$ forms a dense set.

(ii) On account of the second part of the Birkhoff transitivity theorem, whose hypotheses are clearly satisfied, it suffices to verify that for every pair of non-empty open sets U, V, there exists $n \geq 1$ such that

$$\phi^n(U) \cap V \neq \emptyset.$$

For this the lifting $\widehat{\Phi}$ and its interpretation as shift map come in handy. In fact, working on the covering $\mathbb{R}$ and identifying U with an open set inside $[0, 1)$, pick an interval

$$I = (\frac{k}{2^l}, \frac{k+1}{2^l}] \subset U.$$

Here $0 \leq k < 2^l - 1$. Then we can write

$$k = x_0 + x_1 \cdot 2 + \ldots + x_{l-1]} \cdot 2^{l-1},$$

for suitable $x_j \in \{0,1\}$. Then every number x inside I admits a binary expansion

$$x = \big[\frac{x_{l-1}}{2} + \ldots + \frac{x_1}{2^{l-1}} + \frac{x_0}{2^l}\big] + \frac{y_{l+1}}{2^{l+1}} + \frac{y_{l+2}}{2^{l+2}} + \ldots,$$

where our assumption is again that an infinite number of the $y_j \in \{0,1\}$ is non-zero. Keep in mind that the y_j can be varied freely, while the x_j are held fixed.

Applying ϕ on S^1 corresponds to applying $\widehat{\Phi}$ on $\mathbb{R}$, which in turn corresponds to the shift map for the sequence of binary digits. Applying $\widehat{\Phi}$ l times simply eliminates the first l entries and results in

$$\widehat{\Phi}^l(x) = \frac{y_{l+1}}{2^1} + \frac{y_{l+2}}{2^2} + \ldots$$

But the y_{l+j}, $j \geq 1$, can be chosen arbitrarily inside $\{0,1\}$ (up to infinitely many non-vanishing), and this means that

$$\{\widehat{\Phi}^l(x),\, x \in I\} = [0,1).$$

It follows that a fortiori we have

$$\widehat{\Phi}^l\big(U\big) = [0,1).$$

In turn this implies that

$$\phi^l(U) = S^1,$$

which is of course much more than we need.

$\square$