

PROOF OF POINCARÉ-SIEGEL FOR THE SIEGEL CASE

We recall that our goal is to find a function h which is bi-holomorphic around $z = 0$ and such that

$$(0.1) \quad h^{-1} \circ f \circ h = \lambda \cdot z$$

for small enough $|z|$, where the holomorphic f is given by

$$f(z) = \lambda \cdot z + \sum_{k=2}^{\infty} a_k z^k,$$

and moreover $\lambda = e^{2\pi i \alpha}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ a diophantine number. Assume that we can approximately solve this equation so that

$$(0.2) \quad h^{-1} \circ f \circ h = \lambda \cdot z + u,$$

where $u = \sum_{k=2}^{\infty} b_k z^k$. Then we pick w so that

$$(0.3) \quad w(\lambda z) - \lambda \cdot w(z) = u.$$

Since u vanishes quadratically at $z = 0$, given $\varepsilon > 0$ we can pick $r > 0$ such that

$$(0.4) \quad |u'(z)| < \varepsilon, \quad |z| < r.$$

Then we require precise pointwise control of w as well as w' as in the following

Lemma 0.1. *Assuming (0.4), and letting $0 < \rho \leq r$, $0 < \Delta < 1$, we have*

$$\begin{aligned} |w(z)| &< \varepsilon \rho \cdot C(c, d) \cdot \Delta^{-d-1}, \quad |z| \leq \rho \cdot (1 - \Delta), \\ |w'(z)| &< \varepsilon \cdot \frac{C(c, d)}{1 - \Delta} \cdot \Delta^{-d-1}, \quad |z| \leq r \cdot (1 - \Delta). \end{aligned}$$

Proof. We rely on Lemma 4.2 from Lecture 9. Thanks to assumption (0.4), as well as $u(0) = 0$, we infer that

$$|u(z)| < \varepsilon \cdot \rho, \quad |z| \leq \rho.$$

Then Lemma 4.2 implies that

$$|w(z)| < \varepsilon \cdot \rho \cdot C(c, d) \cdot \Delta^{-d-1}, \quad |z| \leq \rho(1 - \Delta).$$

This is the first estimate asserted by the lemma.

For the second bound, observe that if (0.3) holds, then we also have

$$z \lambda w'(\lambda z) - \lambda \cdot z w'(z) = ((\cdot) \cdot w'(\cdot))(\lambda z) - \lambda \cdot z w'(z) = z u'$$

Again applying Lemma 4.2 we infer that

$$|z \cdot w'(z)| \leq \varepsilon \cdot \rho \cdot C(c, d) \cdot \Delta^{-d-1}, \quad |z| \leq \rho(1 - \Delta).$$

In particular, choosing $|z| = \rho(1 - \Delta)$, we find

$$|w'(z)| \leq \varepsilon \cdot \frac{C(c, d)}{1 - \Delta} \cdot \Delta^{-d-1}, \quad |z| = \rho(1 - \Delta).$$

The right hand bound is independent of ρ , so this holds for any $|z| \leq r(1 - \Delta)$. □

Keeping in mind that our goal is to replace (0.2) by an improved version

$$(0.5) \quad (\text{id} + w)^{-1} \circ h^{-1} \circ f \circ h \circ (\text{id} + w) = \lambda \cdot z + u_1,$$

we shall now have to establish two things

- Control the image of both $(\text{id} + w)^{-1}, (\text{id} + w)$ when restricted to small discs.

- Establish a much improved bound for u_1 , and more precisely, that we essentially replace the smallness ε for u as in (0.4) by ε^2 , at the expense of shrinking the disc a bit.

The following lemma takes care of the first point:

Lemma 0.2. *Assuming the smallness conditions*

$$C(c, d) \cdot \varepsilon < \Delta^{d+2}, \quad 0 < \Delta < \frac{1}{4}$$

we have the properties (we use the shorthand $B_r := B_r(0)$)

$$(id + w)(B_{r(1-4\Delta)}) \subset B_{r(1-3\Delta)}, \quad B_{r(1-2\Delta)} \subset (id + w)(B_{r(1-\Delta)}).$$

Here r is such that (0.4) holds.

Proof. Taking advantage of Lemma 0.1, we have for $z \in B_{r(1-4\Delta)}$

$$|(id + w)(z)| \leq |z| + |w(z)| < r(1 - 4\Delta) + \varepsilon r \cdot C(c, d) \cdot \Delta^{-d-1} < r \cdot (1 - 3\Delta)$$

due to our assumption.

For the second part of the lemma, we recall Rouché's theorem from complex analysis, which asserts for our situation that if f, g are two holomorphic functions defined in the neighborhood of $\overline{B_{r(1-\Delta)}}$ and are such that

$$|g(z)| < |f(z)|$$

for all $z \in \partial B_{r(1-\Delta)}$, then the functions

$$f, f + g$$

have the same number of zeroes counted with multiplicity inside $B_{r(1-\Delta)}$. Now pick an arbitrary

$$z_0 \in B_{r(1-2\Delta)}$$

and set

$$f = z - z_0, \quad g = w(z).$$

Then we have

$$|f(z)| \geq r \cdot \Delta, \quad |g(z)| < r \cdot \Delta$$

by our assumption for any $z \in \partial B_{r(1-\Delta)}$, and hence Rouché's theorem applies. This means there is exactly one $z \in B_{r(1-\Delta)}$ with

$$z + w(z) = z_0$$

for each $z_0 \in B_{r(1-2\Delta)}$, proving the lemma. $\square$

The preceding lemma allows us to make sense of

$$(id + w)^{-1} \circ h^{-1} \circ f \circ h \circ (id + w) = (id + w)^{-1} \circ (\lambda \cdot id + u) \circ (id + w)$$

Lemma 0.3. *Assuming the bound (0.4) and further the smallness condition*

$$\varepsilon \cdot C(c, d) < \Delta^{d+2} \cdot (1 - \Delta), \quad 0 < \varepsilon < \Delta < \frac{1}{5},$$

the map

$$(id + w)^{-1} \circ (\lambda \cdot id + u) \circ (id + w) : B_{r(1-4\Delta)} \longrightarrow B_{r(1-\Delta)}$$

is defined and holomorphic.

Proof. From the preceding lemma we know that under the present assumptions, we have

$$(id + w)(B_{r(1-4\Delta)}) \subset B_{r(1-3\Delta)}.$$

Next, we have

$$|(\lambda \cdot id + u)(z)| \leq r(1 - 3\Delta) + \varepsilon \cdot r \leq r(1 - 2\Delta), \quad z \in B_{r(1-3\Delta)}.$$

Again using the preceding lemma, the map

$$(id + w)^{-1} : B_{r(1-2\Delta)} \longrightarrow B_{r(1-\Delta)}$$

is defined and holomorphic. Indeed, observe that $\text{id} + w$ maps $B_{r(1-\Delta)}$ injectively onto its open image $U \supset B_{r(1-2\Delta)}$. To see this, observe that for $z, z' \in B_{r(1-\Delta)}$, we have

$$\begin{aligned} |w(z) - w(z')| &= |(z - z')| \cdot \left| \int_0^1 w'(tz + (1-t)z') dt \right| \\ &\leq \frac{1}{5} \cdot |(z - z')|. \end{aligned}$$

It follows that

$$|(\text{id} + w)(z) - (\text{id} + w)(z')| \geq \frac{4}{5} \cdot |(z - z')|,$$

which implies the injectivity. The fact that there is a holomorphic inverse

$$(\text{id} + w)^{-1} : U \longrightarrow B_{r(1-\Delta)}$$

follows. □

We can then write

$$(\text{id} + w)^{-1} \circ (\lambda \cdot \text{id} + u) \circ (\text{id} + w) = \lambda \cdot z + u_1,$$

and the linchpin is now to establish the following much improved estimate for u_1 :

Lemma 0.4. *Under the same hypotheses as for the previous lemma and also assuming $r \leq 1$, we have the estimate*

$$|u'_1(z)| \leq \varepsilon^2 \cdot \frac{5C(c, d)}{4\Delta^{d+2}}$$

provided $z \in B_{r(1-5\Delta)}$.

Note that we need to estimate the derivative u'_1 since that was our starting point for u . Also, note that we have the much smaller parameter ε^2 here compared to ε before.

Proof. We write the equation defining u_1 in the form

$$(\text{id} + w)(\lambda \cdot z + u_1) = (\lambda \cdot \text{id} + u) \circ (\text{id} + w),$$

which can be written more explicitly as

$$\lambda z + u_1(z) + w(\lambda \cdot z + u_1) = \lambda z + \lambda \cdot w(z) + u(z + w(z))$$

We arrange this in the form of a fixed point equation for u_1 :

$$\begin{aligned} u_1(z) &= -w(\lambda \cdot z + u_1) + \lambda \cdot w(z) + u(z + w(z)) \\ &= -w(\lambda z) + \lambda \cdot w(z) + u(z) + [w(\lambda z) - w(\lambda \cdot z + u_1)] \\ &\quad + [u(z + w(z)) - u(z)] \end{aligned}$$

The sum of the first three terms on the right vanishes thanks to our choice of w . We next estimate the terms in parentheses at the end, for $z \in B_{r(1-4\Delta)}$. Note that by the preceding lemma, we have

$$\lambda z + u_1(z) \in B_{r(1-\Delta)}$$

for such z , and further any point on the straight line segment linking λz to $\lambda z + u_1(z)$ is in $B_{r(1-\Delta)}$. We conclude that

$$\begin{aligned} |w(\lambda z) - w(\lambda \cdot z + u_1)| &= \left| \int_0^1 w'(\lambda \cdot z + t \cdot u_1) \cdot u_1(z) dt \right| \\ &\leq |u_1(z)| \cdot \varepsilon \cdot \frac{C(c, d)}{1 - \Delta} \cdot \Delta^{-d-1}, \end{aligned}$$

where we have taken advantage of Lemma 0.1. Our assumptions then imply that the preceding can be bounded by

$$|u_1(z)| \cdot \varepsilon \cdot \frac{C(c, d)}{1 - \Delta} \cdot \Delta^{-d-1} \leq \frac{|u_1(z)|}{5}.$$

We further have the estimate

$$\begin{aligned} \left| u(z + w(z)) - u(z) \right| &\leq |w(z)| \cdot \int_0^1 |u'(z + tw(z))| dt \\ &\leq \varepsilon r \cdot C(c, d) \cdot \Delta^{-d-1} \cdot \varepsilon. \end{aligned}$$

Here we have again used Lemma 0.1. Also recalling the assumption $r \leq 1$, we then find

$$|u_1(z)| \leq \frac{5}{4} \cdot C(c, d) \cdot \Delta^{-d-1} \cdot \varepsilon^2, \quad z \in B_{r(1-4\Delta)}.$$

Invoking the Cauchy formula, we can infer a derivative bound from this at the expense of shrinking the domain to $B_{r(1-5\Delta)}$:

$$|u'_1(z)| \leq \frac{5}{4} \cdot C(c, d) \cdot \Delta^{-d-2} \cdot \varepsilon^2, \quad z \in B_{r(1-5\Delta)}.$$

□

Let us summarise what we have achieved thus far: starting from the assumption (0.4) and the relation (0.2), we have constructed a slightly modified

$$\tilde{h} = h \circ (\text{id} + w),$$

such that on the slightly smaller disc

$$B_{r(1-5\Delta)}$$

for some $0 < \Delta < \frac{1}{5}$, we have the relation

$$\tilde{h}^{-1} \circ f \circ \tilde{h} = \lambda z + u_1,$$

where now u_1 satisfies the estimate stated in the preceding lemma. This estimate is much sharper than the one for u' , provided ε is small enough in relation to Δ .

But now we can re-start the whole process, with r replaced by $r(1 - 5\Delta)$ and u replaced by u_1 . We then simply need to choose the parameters suitably that the whole process converges (very rapidly!) to the desired conjugation map. For this, we make the following choices:

$$(0.6) \quad r_n = \frac{r}{2} \cdot (1 + 2^{-n})$$

where $r \leq 1$ is chosen as in (0.4) with $\varepsilon = \varepsilon_0$ sufficiently small, as determined below. Next, we set

$$(0.7) \quad \Delta_n = \frac{1}{10(2^n + 1)}, \quad n \geq 1,$$

which implies that

$$\frac{r_{n+1}}{r_n} = \frac{1 + 2^{-n-1}}{1 + 2^{-n}} = 1 - \frac{2^{-n-1}}{1 + 2^{-n}} = 1 - 5\Delta_n.$$

Now assuming that

$$|u'_n| < \varepsilon_n$$

on B_{r_n} with $n \geq 1$, then we get from the preceding lemma and the corresponding choice of w_n that

$$|u'_{n+1}(z)| \leq \varepsilon_n^2 \cdot \frac{5C(c, d)}{4\Delta_n^{d+2}} \leq C_1(c, d) \cdot (2^n + 1)^{d+2} \cdot \varepsilon_n^2, \quad z \in B_{r_{n+1}}(0).$$

Thus we can set

$$\varepsilon_{n+1} = C_1(c, d) \cdot (2^n + 1)^{d+2} \cdot \varepsilon_n^2.$$

It follows that if we introduce the new sequence

$$\gamma_n := C_2(c, d) \cdot (2^n + 1)^{d+2} \cdot \varepsilon_n,$$

where C_2 is chosen so that

$$C_2 = M \cdot (C_1 + 1) \cdot 2^{d+2},$$

with $M \geq 1$ a large enough constant which we will chosen below, then we find that

$$\begin{aligned}\gamma_{n+1} &= C_1(c, d) \cdot (2^n + 1)^{d+2} \cdot (2^{n+1} + 1)^{d+2} \cdot C_2 \cdot \varepsilon_n^2 \\ &= C_2^{-1} \cdot C_1(c, d) \cdot \frac{(2^{n+1} + 1)^{d+2}}{(2^n + 1)^{d+2}} \cdot \gamma_n^2 \\ &\leq \gamma_n^2.\end{aligned}$$

In particular if we pick the smallness constant ε_0 such that

$$(0.8) \quad C_2(c, d) \cdot (2^0 + 1)^{d+2} \cdot \varepsilon_0 < \frac{1}{2},$$

then the γ_n and a fortiori the ε_n will converge faster than exponentially toward zero. Specifically, we obtain that

$$(0.9) \quad |\gamma_n| \leq 2^{-2^n}, |\varepsilon_n| \leq C_2^{-1} \cdot 2^{-2^n} (2^n + 1)^{-d-2} \leq M^{-1} \cdot 2^{-2^n} (2^n + 1)^{-d-2}.$$

We observe that if M is chosen sufficiently large, then the smallness condition of Lemma 0.3 is automatically satisfied for $\triangle_n, \varepsilon, n \geq 0$. From Lemma 0.1 we deduce that

$$|w_n(z)| \leq C_3 \cdot C_4^n \cdot 2^{-2^n}$$

for $z \in B_{r_{n+1}}(0)$ and suitable constants $C_{3,4}$.

The conjugating map h which achieves

$$h^{-1} \circ f \circ h(z) = \lambda \cdot z$$

on $B_{\frac{r}{2}}(0)$ is then given by

$$h = \lim_{n \rightarrow \infty} (\text{id} + w_1) \circ (\text{id} + w_2) \circ \dots \circ (\text{id} + w_n)$$

That this indeed converges on $B_{\frac{r}{2}}(0)$ follows by observing that (using Lemma 0.2)

$$(0.10) \quad (\text{id} + w_1) \circ (\text{id} + w_2) \circ \dots \circ (\text{id} + w_n) \Big|_{B_{r_{n+1}}(0)} : B_{r_{n+1}}(0) \longrightarrow B_r(0)$$

But then calling

$$h_n := (\text{id} + w_1) \circ (\text{id} + w_2) \circ \dots \circ (\text{id} + w_n),$$

we infer that

$$\begin{aligned}|h_{n+1} - h_n|(z) &= |h_n \circ (\text{id} + w_{n+1})(z) - h_n(z)| \\ &\leq |w_{n+1}(z)| \cdot \sup_{z' \in B_{r_{n+1}}(1-3\triangle_{n+1})} |h'_n(z')|,\end{aligned}$$

provided $z \in B_{\frac{r}{2}} \subset B_{r_{n+2}}$. But then using the Cauchy integral formula we deduce that

$$\sup_{z' \in B_{r_{n+1}}(1-3\triangle_{n+1})} |h'_n(z')| \leq C(r, d) \cdot \triangle_{n+1}^{-1} \leq C(r, d) \cdot D^n,$$

for suitable constants $C(r, d), D$. Thanks to the rapid decrease of the w_n , we then deduce that

$$|h_{n+1} - h_n|(z) \leq 2^{-(\frac{3}{2})^n}, z \in B_{\frac{r}{2}}(0),$$

for large enough n , which implies the convergence of the h_n on $B_{\frac{r}{2}}(0)$.

Since

$$h_n^{-1} \circ f \circ h_n = \lambda \cdot z + u_n, z \in B_{\frac{r}{2}}(0)$$

by construction and $|u_n| \rightarrow 0$ rapidly there, we obtain that

$$h^{-1} \circ f \circ h(z) = \lambda \cdot z, z \in B_{\frac{r}{2}}(0),$$

as desired.