DIFFEOMORPHISMS AROUND 'ELLIPTIC' FIXED POINTS: THE POINCARÉ-SIEGEL THEOREM

We now try to approach the question of how to deal with the structure of diffeomorphisms around 'elliptic fixed points'. Precisely, let

$$f \in \mathrm{Diff}^1(\mathbb{R}^n)$$

with fixed point $x_0 = 0$, i. e. $f(0) = 0$. Moreover, assume that $Df(0)$ is not hyperbolic, in the sense that there are eigenvalues with modulus $|\lambda| = 1$. We again would like to conjugate such a map to its linearisation. However, by contrast to the situation of Hartman-Grobman, this is much harder and there are subtle obstacles. In fact, to arrive at a satisfactory result, we shall have to very much specialise the situation, and only consider the case $n = 2$. Furthermore, it will be crucial to impose a much more rigid *analytic structure* on f. Our discussion in the following follows the one in Hasselblatt-Katok, which in turn follows the approach developed by Kolmogorov-Arnold-Moser(KAM).

1. Holomorphic maps around $z_0 = 0 \in \mathbb{C}$ and analytic conjugations

Identifying $\mathbb{R}^2 \simeq \mathbb{C}$, we now let

$$f : U \longrightarrow \mathbb{C}$$

a holomorphic (complex analytic) map from an open neighbourhood U of $z = 0$ to the complex plane. We assume $f(0) = 0$. More precisely, we shall assume that

$$f(z) = \lambda \cdot z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which converges absolutely for some $|z| < r_*, r_* > 0$, and such that

$$|\lambda| = 1.$$

Then we pose the following

Question: *Does there exist an analytic local homeomorphism*

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n,$$

such that

$$h^{-1} \circ f \circ h(z) = \lambda \cdot z? \tag{1.2}$$

The answer in this generality is clearly *no*, since if $\lambda = 1$, the preceding relation would imply that

$$f(z) = z.$$

Furthermore, if $\lambda = e^{2\pi i \frac{p}{q}}$, where $\frac{p}{q}$ is a rational number in reduced form, then *generically* we cannot hope to be able to find such h. For example, let $\lambda = e^{\pi i} = -1$. Then setting

$$f(z) = -z + \sum_{n \geq 2} a_n z^n,$$

if analytic h with the desired conjugation property exists, we have

$$f^{(2)}(z) = f \circ f(z) = z.$$

Inserting the power series, this results in

$$z - \sum_{n \geq 2} a_n z^n + a_2 \cdot \big(- z + a_2 z^2 + \dots \big)^2 + a_3 \cdot \big(- z + a_2 z^2 + \dots \big)^3 + o\big(|z|^3\big) = z$$

Testing at the level of third powers, this results in

$$-2a_3 - 2a_2^2 = 0,$$

which imposes a co-dimension one condition. Nonetheless there is a non-trivial holomorphic function f defined around $z = 0$ with

$$f \circ f = z,$$

namely

$$f(z) = \frac{-z}{1-z}.$$

In light of these observations, we now restrict to $\lambda = e^{2\pi i\alpha}$ with *irrational* $\alpha \in \mathbb{R}$. Can the previously posed question be answered in that case?

It turns out that the optimal result is still extraordinarily subtle, in that the conjugation exists for 'most' irrational α, but not all. It was only completely resolved by the 1980s, in work that resulted in the Fields medal for J.-C. Yoccoz.
Our approach shall be to take advantage of a KAM type scheme, which is useful in many other types of contexts. The earliest approach due to C.-L. Siegel in the 1940s did not use a KAM scheme, and it is actually his approach which ultimately led to suggest the optimal result finally proved by Yoccoz. The name Poincaré occurs in the title since he considered the 'hyperbolic case' $|\lambda| \neq 1$, $\lambda \neq 0$, for which he established the existence of an analytic h that achieves the desired conjugation. No KAM scheme is required to handle this case.

2. Diophantine irrationals

Our goal in the sequel shall be to show that a holomorphic h achieving (1.2) exists for *arbitrary* holomorphic f of the form (1.1), and provided we work in a neighborhood sufficiently close to $z = 0$, under the condition that

$$\lambda = e^{2\pi i\alpha}$$

and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ cannot be too well approximated by rational numbers in the following sense

Definition 2.1. *We say that $\alpha \in \mathbb{R}$ is diophantine, provided there exist numbers $c > 0, d > 1$ with the property that*

$$\big|p\alpha - q\big| > c|p|^{-d}$$

for any pair of integers (p, q) with $p \neq 0$.

These numbers are generic in a sense, since their complement can be shown to have measure zero. We observe right away that

Lemma 2.2. *If α is diophantine, then $\lambda = e^{2\pi i\alpha}$ satisfies*

$$\big|\lambda^p - 1\big| \geq c_1 \cdot |p|^{-d},\ p \in \mathbb{Z}\backslash\{0\}$$

for suitable $c_1 > 0, d > 1$.

Proof. Given $p \in \mathbb{Z}\backslash\{0\}$, choose $q \in \mathbb{Z}$ with the property that

$$0 < \big|\alpha p - q\big| < 1.$$

By hypothesis we have that

$$\big|\alpha p - q\big| \geq c \cdot |p|^{-d},\ \big|\alpha p - q \pm 1\big| \geq c \cdot |p|^{-d}$$

But then we have

$$\big|\lambda^p - 1\big| = \big|e^{2\pi i\alpha\cdot p} - e^{2\pi iq}\big| = \big|e^{2\pi i(\alpha\cdot p - q)} - 1\big| \geq C \cdot c \cdot |p|^{-d}$$

for a suitable constant $C > 0$, whence the conclusion with $c_1 = C \cdot c$, as desired. □

Our goal in the sequel shall be to prove the following theorem

Theorem 2.3. *(Siegel part of Poincaré-Siegel). Let $\lambda = e^{2\pi i\alpha}$ with α diophantine. Given f holomorphic around $z = 0$ and of the form* (1.1)*, there exist $\delta > 0$ and a bi-holomorphic*[1] *h defined near $z = 0$ such that*
$$h^{-1} \circ f \circ h(z) = \lambda \cdot z,\ |z| < \delta.$$

3. The basic paradigm for finding h: a Newton iteration scheme

Our goal is to construct h such that
$$h^{-1} \circ f \circ h = \lambda \cdot z$$
for $|z|$ small enough. In fact, we will achieve this via an inductive procedure, constructing better and better approximations for h. Let us assume that we have a h such that
$$h^{-1} \circ f \circ h = \lambda \cdot z + u$$
where u is a holomorphic function which vanishes at least quadratically at $z = 0$. We can then try to 'improve h' and decrease this error u further. For this we shall replace h by (here $\mathrm{id}(z) = z$)
$$h \circ \big(\mathrm{id} + w\big)$$
for some holomorphic w which vanishes quadratically at $z = 0$. Consider hence
$$\big(\mathrm{id} + w\big)^{-1} \circ h^{-1} \circ f \circ h \circ \big(\mathrm{id} + w\big) = \big(\mathrm{id} + w\big)^{-1} \circ \big(\lambda \cdot z + u\big) \circ \big(\mathrm{id} + w\big). \tag{3.1}$$
The goal here is to pick w in such a way that the right hand side is closer to $\lambda \cdot z$ than the expression $\lambda \cdot z + u$.

The first order of the day is to simplify the function $\big(\mathrm{id} + w\big)^{-1}$. For this observe that
$$\big(\mathrm{id} + w\big) \circ \big(\mathrm{id} - w\big)(z) = z - w(z) + w\big(z - w(z)\big).$$
Further we have
$$-w(z) + w\big(z - w(z)\big) = -\int_0^1 w(z) \cdot w'(z - tw(z))\, dt,$$
and so this is *quadratic in w*. This means that up to a quadratic error in w, which we label $E_2(w)$, we may replace (3.1) by the following
$$\big(\mathrm{id} + w\big)^{-1} \circ h^{-1} \circ f \circ h \circ \big(\mathrm{id} + w\big) = \big(\mathrm{id} - w\big) \circ \big(\lambda \cdot z + u\big) \circ \big(\mathrm{id} + w\big) + E_2(w). \tag{3.2}$$
Let us compute the expression on the right explicitly at z: this becomes
$$\begin{aligned}&\big(\mathrm{id} - w\big) \circ \big(\lambda \cdot z + u\big) \circ \big(\mathrm{id} + w\big)(z)\\ &= \lambda z + \lambda \cdot w(z) + u\big(z + w(z)\big) - w\big(\lambda z + \lambda w(z) + u(z + w(z))\big).\end{aligned}$$
This looks complicated but can be simplified a lot if we neglect quadratic terms, by which we mean terms which can be expressed like before in terms of products of two factors w, u or their derivatives. This allows us to make the following replacements:
$$u\big(z + w(z)\big) \Longrightarrow u(z),\ w\big(\lambda z + \lambda w(z) + u(z + w(z))\big) \Longrightarrow w\big(\lambda z\big). \tag{3.3}$$
We then arrive at the important identity
$$\big(\mathrm{id} + w\big)^{-1} \circ h^{-1} \circ f \circ h \circ \big(\mathrm{id} + w\big) = \lambda z + \lambda \cdot w(z) - w\big(\lambda z\big) + u(z) + E_3(u, w). \tag{3.4}$$
Here the final term $E_3(u, w)$ again denotes terms quadratic in u, w and their derivatives.

At this stage, we can see which choice of w is going to 'decimate' the error u, up to much smaller error terms. We need to impose the condition
$$\boxed{w\big(\lambda z\big) - \lambda \cdot w(z) = u(z)} \tag{3.5}$$
provided $|z|$ is small enough.
The Diophantine nature of α will be crucial to obtain reasonable bounds for the solution of this equation,

[1]By this we mean that there are open neighborhoods U, V of $z = 0$ such that $h : U \to V$ admits a holomorphic inverse $h^{-1} : V \to U$.

which is *degenerate* in a certain sense. In fact, one encounters here the so-called *small divisor phenomenon*, as we shall see shortly.

4. Bounds for the solution of (3.5)

The preceding discussion allowed us to reduce the construction of the inductive correction w to solving (3.5), but at the price of having to differentiate w for the error term. Performing infinitely many correction steps implies a kind of 'infinite derivative loss', which usually dooms an iterative scheme. The remarkable thing about working with analytic functions is that one can control the derivatives of such functions in terms of the undifferentiated functions, albeit at the expense of modifying the domain a bit. We begin by providing some basic tools from complex analysis, namely how to infer control of its Taylor coefficients, and vice versa:

Lemma 4.1. *(1) Let*

$$\phi(z) = \sum_{k=0}^{\infty} a_k z^k$$

a function which is complex analytic on $B_r(0)$ and continuous on $\overline{B_r(0)}$. Then we have

$$\big|a_k\big| \leq r^{-k} \cdot \big\|\phi\big\|_{L^\infty(B_r(0))}.$$

(2) Assume conversely that ϕ is formally given by the above formula and that the coefficients a_k satisfy the bounds

$$\big|a_k\big| \leq K \cdot r^{-k}$$

for some $K \geq 0$. Then the function ϕ is complex analytic on $B_r(0)$, and we have the estimate

$$\big\|\phi\big\|_{L^\infty(B_{r-\delta}(0))} \leq K r \delta^{-1}$$

provided $0 < \delta < r$.

Proof. (1) By Cauchy's integral formula we have

$$a_k = \frac{1}{2\pi i} \cdot \int_{\partial B_r(0)} \frac{\phi(z)}{z^{k+1}}\, dz.$$

But then we infer that

$$\big|a_k\big| \leq \frac{1}{2\pi} \cdot 2\pi r \cdot \big\|\phi\big\|_{L^\infty(B_r(0))} \cdot r^{-k-1} = r^{-k} \cdot \big\|\phi\big\|_{L^\infty(B_r(0))}.$$

(2) It suffices to estimate

$$\sum_{k=0}^{\infty} \big|a_k\big| \cdot (r-\delta)^k \leq K \cdot \sum_{k=0}^{\infty} (1 - \frac{\delta}{r})^k \leq K \cdot \frac{r}{\delta}.$$

This establishes the absolute convergence of the sum defining ϕ on $B_r(0)$ as well as the estimate. □

Let us now see how to solve (3.5) in terms of power series. Assume that

$$u(z) = \sum_{k=2}^{\infty} a_k z^k$$

Further, assume that

$$w(z) = \sum_{k=2}^{\infty} b_k z^k.$$

Then we can formally expand

$$w\big(\lambda z\big) - \lambda \cdot w(z) = \sum_{k=2}^{\infty} b_k \cdot (\lambda^k - \lambda) \cdot z^k$$

Thus if (3.5) admits a complex analytic solution, it has to be of the form

$$w(z) = \sum_{k=2}^{\infty} \frac{a_k}{\lambda^k - \lambda} \cdot z^k. \tag{4.1}$$

To see that this is indeed a well-behaved analytic function, we have

Lemma 4.2. *Assume that u is analytic on $B_r(0)$, $r > 0$, with $\|\phi\|_{L^\infty(B_r(0)} < \infty$. Further assume that $\lambda = e^{2\pi i\alpha}$ with $\alpha \in \mathbb{R}$ diophantine in the sense of Definition 2.1. Then the function defined by (4.1) is complex analytic on $B_r(0)$, and satisfies a quantitative bound on a slightly smaller disc $B_{r(1-\triangle)}$ with $1 > \triangle > 0$, namely*

$$\big|w(z)\big| < \big\|u\big\|_{L^\infty(B_r(0)} \cdot C(c,d) \cdot \triangle^{-d-1},$$

provided $z \in B_{r(1-\triangle)}$. Here c, d are like in the definition of diophantine α.

Proof. From the previous lemma, we know that

$$\big|a_k\big| \leq r^{-k} \cdot \big\|u\big\|_{L^\infty(B_r(0))}.$$

Then Lemma 2.2 implies that

$$\Big|\frac{a_k}{\lambda^k - \lambda}\Big| \leq r^{-k} \cdot C_1(c) \cdot k^d \cdot \big\|u\big\|_{L^\infty(B_r(0))}.$$

Now observe that for $|\zeta| < 1$, we have that

$$\big|\sum_{k=0}^{\infty} k^d \cdot \zeta^k\big| \leq \sum_{k=0}^{\infty} k^d \cdot |\zeta|^k \leq C(d) \cdot \big(1 - |\zeta|\big)^{-d-1}$$

for arbitrary $d \geq 0$ and a suitable constant $C(d)$. *Exercise: show this!*
We conclude that if $|z| \leq (1 - \triangle) \cdot r$, $\triangle > 0$, we have

$$\begin{aligned}\Big|\sum_{k=2}^{\infty} \frac{a_k}{\lambda^k - \lambda} \cdot z^k\Big| &\leq C_1(c) \cdot \big\|u\big\|_{L^\infty(B_r(0))} \cdot \sum_{k=2}^{\infty} k^d \cdot \big(1 - \triangle\big)^k \\ &\leq C_1(c) \cdot C(d) \cdot \triangle^{-d-1} \cdot \big\|u\big\|_{L^\infty(B_r(0))}.\end{aligned}$$

This implies that the series representing w converges absolutely inside $B_r(0)$ and hence defines a complex analytic function there. This function satisfies the claimed bound. □

We see that solving the equation (3.5) is possible on $B_r(0)$ but if we insist on quantitative bounds, we have to shrink the disc $B_r(0)$ we work on. Iterating this process will result in larger and larger 'losses'

$$\big(1 - \triangle\big)^{-d-1}$$

if we want the process to converge to some disc $B_{r_*}(0)$ with $r_* > 0$. The key feature that will make this work is that the sequence of functions u to which we apply the preceding lemma shrinks extremely fast to zero, and this will more than compensate for the larger and larger losses incurred due to picking smaller and smaller constants $\triangle > 0$.
Observe that the factor $\lambda^k - \lambda$ is the 'small divisor' mentioned earlier.