TOPICS IN PROBABILITY. PART II: UNIVERSALITY

Exercise sheet 9: Universality and semicircle law

Exercise 1 (Moments of semicircle law)**.** *;*

Let μ be the semicircle law, i.e., $\mu(\mathrm{d}x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{\{|x|\le 2\}}\mathrm{d}x$. Show that for $k \in \mathbb{N}_0$,

$$\mu[x^k] = \begin{cases} 0 & k \text{ odd;} \\ C_{k/2} & k \text{ even,} \end{cases}$$

where $C_k = \frac{1}{1+k}\binom{2k}{k}$ is the kth Catalan number.

Proof. Note that μ is symmetric, supported on a compact set, thus, all the moments exist and all odd moments are zero. Let $k \in \mathbb{N}$ be even, with the substitute $x = 2\sin(y)$ we obtain,

$$\begin{aligned}\mu[x^k] &= \frac{1}{\pi}\int_0^2 x^k\sqrt{4-x^2}\mathrm{d}x = \frac{2^{k+2}}{\pi}\int_0^{\pi/2}\sin(y)^k\cos^2(y)\mathrm{d}y \\ &= \frac{2^{k+2}}{\pi}\left(\int_0^{\pi/2}\sin(y)^k\mathrm{d}y - \int_0^{\pi/2}\sin(y)^{k+2}\mathrm{d}y\right).\end{aligned}$$

Using integration by parts formula, one easily obtains that $\int_0^{\pi/2}\sin(y)^k\mathrm{d}y = \frac{k-1}{k}\int_0^{\pi/2}\sin(y)^{k-2}\mathrm{d}y$. And iteratively, $\int_0^{\pi/2}\sin(y)^k\mathrm{d}y = \frac{(k-1)!!}{k!!}\frac{\pi}{2}$. Therefore,

$$\begin{aligned}\mu[x^k] &= 2^{k+1}\frac{(k-1)!!}{k!!}\left(1-\frac{k+1}{k+2}\right) = 2^{k+1}\frac{(k-1)!!}{(k+2)!!} = \frac{2^{k+1}}{k+2}\frac{k!}{(2^{k/2}(k/2)!)^2} \\ &= \frac{1}{1+k/2}\binom{k}{k/2} = C_{k/2}.\end{aligned}$$

$\square$

Exercise 2 (Trace, operator and Frobenius norms)**.** *;*

Let $(A^i)_{i=1}^k$ be matrices of sizes $m_{i-1}\times m_i$ such that $m_0 = m_k$. Show that for any $1 \le i < j \le k$,

$$|\mathrm{Trace}(A^1A^2\dots A^k)| \le \|A^i\|_F\|A^j\|_F\prod_{l:l\ne i,j}\|A^l\|_{op},$$

where $\|A\|_F = \sqrt{\mathrm{Trace}(\bar{A}^tA)} = \sqrt{\sum_{i,j}|a_{ij}|^2}$ is the Frobenius norm and $\|A\|_{op} = \sup_{\|x\|_2=1}\|Ax\|_2$ the operator norm.

Proof. Recall that $\mathrm{Trace}(AB) = \mathrm{Trace}(BA)$ and observe that $|\mathrm{Trace}(AB)| \le \|A\|_F\|B\|_F$ (by Cauchy-Schwarz for vectors $(a_{ij})_{i,j}$ and $(b_{ij})_{ij}$ in $\mathbb{R}^{mn}$ if A is an $m\times n$-matrix). Furthermore, it holds $\|AB\|_F \leq \|A\|_{op}\|B\|_F$. Indeed, write $b_1, \ldots, b_m$ for the column vectors of B, then $AB = (Ab_1, \ldots, Ab_m)$ and by definition of the Frobenius norm,

$$\|AB\|_F^2 = \sum_{k=1}^m \|Ab_k\|_2^2 \leq \|A\|_{op}^2 \sum_{k=1}^m \|b_k\|_2^2 = \|A\|_{op}^2 \|B\|_F^2.$$

In particular, we obtain,

$$\begin{aligned}
|\mathrm{Trace}(A^1 A^2 \ldots A^k)| &= |\mathrm{Trace}(A^{j+1} \ldots A^k A^1 \ldots A^i \ldots A^j)| \\
&\leq \|A^{j+1} \ldots A^k A^1 \ldots A^i\|_F \|A^{i+1} \ldots A^j\|_F \\
&\leq \|A^{j+1} \ldots A^k A^1 \ldots A^{i-1}\|_{op} \|A^i\|_F \|A^{i+1} \ldots A^{j-1}\|_{op} \|A^j\|_F \\
&\leq \prod_{l \neq i,j} \|A^l\|_{op} \|A^i\|_F \|A^j\|_F.
\end{aligned}$$

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Exercise 3 (Bound on difference of eigenvalues)**.** *;*

Let A, B be two Hermitian $n \times n$ matrices. Let us denote their eigenvalues $(\lambda_i^A)_{i=1}^n$ and $(\lambda_i^B)_i$, respectively. Show that

$$\inf_\sigma \sum_{i=1}^n |\lambda_i^A - \lambda_{\sigma(i)}^B|^2 \leq \|A - B\|_F^2,$$

where the infimum is taken over all permutations of n elements.

Proof. Let us start by showing that for any Hermitian matrix M (or more generally, any $n \times n$ complex valued matrix), $\sum_{k=1}^n |\lambda_k|^2 \leq \|M\|_F^2$. Indeed, by Schur decomposition, there exist a unitary matrix Q and an upper-triangular matrix U with the eigenvalues of M on its diagonal such that $M = QUQ^{-1}$ (in the case of Hermitian matrix, one can find unitary Q such that U is diagonal with real entries). Note further that $\|QA\|_F^2 = \mathrm{Trace}((\overline{QA})^t QA) = \mathrm{Trace}(\bar{A}^t \bar{Q}^t QA) = \mathrm{Trace}(\bar{A}^t A) = \|A\|_F^2$ since Q is unitary; and analogously, $\|A\|_F = \|AQ^{-1}\|_F$. Therefore,

$$\|M\|_F^2 = \|QUQ^{-1}\|_F^2 = \|U\|_F^2 \geq \sum_{i=1}^n |\lambda_i|^2.$$

Note that if M is Hermitian, we get $\|M\|_F^2 = \sum_{i=1}^n \lambda_i^2$ (by the above observation in parenthesis). This immediately implies that to prove the desired claim for Hermitian A, B, it suffices to show that $\sup_\sigma \sum_{i=1}^n \lambda_i^A \lambda_{\sigma(i)}^B \geq \langle A, B \rangle_F (= \mathrm{Trace}(\bar{A}^t B) = \mathrm{Trace}(AB))$. Since both A and B are Hermitian, there exist unitary U, Q such that $A = QD_A\bar{Q}^t$, $B = UD_B\bar{U}^t$ for diagonal matrices D_A, D_B containing (real) eigenvalues of A and B, respectively. Note further that $\bar{Q}^t BQ$ has the same eigenvalues as B (if v is an eigenvector corresponding to EV λ^B of B, then $\bar{Q}^t v$ is the eigenvector of $\bar{Q}^t BQ$ corresponding to λ^B). Together with properties of the trace, this implies that $\langle A, B \rangle_F = \sum_{i=1}^n \lambda_i^A (\bar{Q}^t BQ)_{ii} = \sum_i \lambda_i^A \sum_j \lambda_j^B |v_{ij}|^2 = \langle \lambda^A, W\lambda^B \rangle$, where $V = (v_{ij}) = \bar{Q}^t U$ (again unitary) and $W = (|v_{ij}|^2)_{ij}$. Note that W is doubly stochastic, meaning that the sum of the elements of any of its rows and columns is one. By Birkhoff-von Neumann theorem, there exist permutation matrices $(P_k)_{k=1}^m$ (for some m) and

$a_k \geq 0$ such that $\sum_{k=1}^m a_k = 1$ and $W = \sum_{k=1}^m a_k P_k$. Therefore, as desired

$$\langle A, B\rangle_F = \langle \lambda^A, W\lambda^B\rangle = \sum_{k=1}^m a_k \underbrace{\langle \lambda^A, P_k\lambda^B\rangle}_{\leq \sup_\sigma \sum_{i=1}^n \lambda_i^A \lambda_{\sigma(i)}^B} \leq \sup_\sigma \sum_{i=1}^n \lambda_i^A \lambda_{\sigma(i)}^B.$$

Remark: there is an alternative proof that can be found in Terence Tao's blog.

□

Exercise 4 (Bound on the operator norm of Wigner matrix with bounded entries)**.** *;*
Let X be a Wigner $N \times N$ matrix with mean-zero, uniformly bounded (wlog, by one) entries, i.e., $(X_{ij})_{i\leq j\leq N}$ are independent with $\mathbb{E}[X_{ij}] = 0$ and $|X_{ij}| \leq 1$ almost surely and $X_{ji} = \bar{X}_{ij}$. Show that there exists $C, c > 0$ (absolute constants independent of N) such that $\mathbb{P}[\|X\|_{op}/\sqrt{N} \geq t] \leq e^{-cNt^2}$ for all $t \geq C$.
You may proceed as follows:

1. *Show (using an appropriate concentration of measure inequality you know) that for any unit vector $x \in \mathbb{C}^N$ and $N \times N$-matrix $\tilde{X}$ with (all!) mutually independent, uniformly bounded by one entries of mean zero, $\mathbb{P}[|\tilde{X}x|/\sqrt{N} > t] \leq e^{-cNt^2}$ for all $t \geq C$ for some appropriate absolute constant $C > 0$;*
2. *Prove that $\mathbb{P}[\|\tilde{X}\|_{op}/\sqrt{N} > t] \leq \mathbb{P}[\bigcup_{x\in G}\{|\tilde{X}x|/\sqrt{N} > t/2\}]$ for a maximal $1/2$-net $G \subset \mathbb{C}^N \cap \mathbb{S}^{2N-1}$, i.e., all points of G are separated by a distance at least $1/2$, and G is maximal w.r.t. set-inclusion;*
3. *Estimate the number of points of G appropriately and conclude the result for $\|\tilde{X}\|_{op}/\sqrt{N}$;*
4. *Conclude for $\|X\|_{op}/\sqrt{N}$.*

Proof. (1): Let $x = (x_1, \ldots, x_n) \in \mathbb{C}^N$ be a unit vector, i.e., $\sum_{i=1}^N |x_i|^2$, and $\tilde{X}$ any $N \times N$-matrix with independent entries of mean zero and that are uniformly bounded by one. Let us denote its rows by $\tilde{X}^i$. By Azuma-Hoeffding inequality (applied to real and imaginary parts of $\pm X^i \cdot x$), for any $u \geq 0$,

$$\mathbb{P}[|\tilde{X}^i \cdot x| \geq u] \leq 4e^{-cu^2/\sum_{i=1}^N |x_i|^2} = 4e^{-cu^2}$$

for an appropriate $c > 0$. Therefore, using equivalent characterizations of sub-gaussian law, $\mathbb{E}[e^{c'|\tilde{X}^i\cdot x|^2}] \leq C$ for appropriate $c', C > 0$ (absolute constants depending only on the above c). This, in turn, implies that $\mathbb{E}[e^{c'|\tilde{X}x|^2}] \leq C^N$, and thus, by Markov's inequality,

$$\mathbb{P}[|\tilde{X}x|/\sqrt{N} > t] \leq C^N e^{-(c')^2Nt^2} \leq e^{-(c')^2Nt^2/2}$$

for all $t \geq 0 \vee (2\log(C)/(c')^2)$.

(2): Note that $\|\tilde{X}\|_{op} = \sup_{x\in\mathbb{C}^n, \|x\|=1} |\tilde{X}x|$, and hence,

$$\{\|\tilde{X}\|_{op} \geq t\sqrt{N}\} = \bigcup_{x\in\mathbb{C}^n, \|x\|=1} \{|\tilde{X}x| \geq t\sqrt{N}\}.$$

By compactness of the sphere, there exists unit vector x_0 such that the above supremum is attained at x_0. Either $x_0 \in G$, or there exists $y \in G$ such that $|x - y| \leq 1/2$ (by maximality of G). Note that in the latter case, since $|\tilde{X}(x - y)| \leq \|\tilde{X}\|_{op}/2$, by triangle inequality, $|\tilde{X}y| \geq \|\tilde{X}\|_{op}/2$. This implies that, if $\|\tilde{X}\|_{op} \geq t\sqrt{N}$, then there exists $y \in G$ such that

$|\tilde{X}y| > \sqrt{N}t/2$, and so, for a maximal 1/2-net $G \subset \mathbb{C}^N \cap \mathbb{S}^{2N-1}$,

$$\mathbb{P}[\|\tilde{X}\|_{op}/\sqrt{N} > t] \leq \mathbb{P}[\bigcup_{x\in G}\{|\tilde{X}x|/\sqrt{N} > t/2\}].$$

(3): We now want to show that the cardinality of G is bounded by K^N for an appropriate absolute constant $K > 0$. To this end, consider 1/4-balls centered at points of G. By definition of G, these are disjoint. On the other hand, their union is contained in the ball of radius 3/2 centered at 0. Therefore, the cardinality of G is bounded by the ratio of the volumes of the latter ball and a 1/4-ball. This number is clearly bounded by K^N for an appropriate $K > 4$. Together with the previous parts, we obtain

$$\mathbb{P}[\|\tilde{X}\|_{op}/\sqrt{N} > t] \leq K^N e^{-(c')^2 N t^2/2} \leq e^{-(c')^2 N t^2/4}$$

for all $t \geq 0 \vee (\frac{4}{(c')^2}\log(K \vee C))$.

(4): Let X be the desired Wigner matrix. Note that $\tilde{X}$ defined by $\tilde{X}_{ij} = X_{ij}$ for $i \leq j$ and 0 otherwise and $\tilde{X}'$ by $\tilde{X}'_{ij} = X_{ij}$ for $i > j$ and 0 otherwise, both satisfy assumptions of the previous parts. Thus, for appropriate $c, C > 0$

$$\mathbb{P}[\|X\|_{op}/\sqrt{N} > t] \leq \mathbb{P}[\|\tilde{X}\|_{op}/\sqrt{N} > t/2] + \mathbb{P}[\|\tilde{X}'\|_{op}/\sqrt{N} > t/2] \leq e^{-cNt^2}$$

for all $t \geq C$. $\square$