TOPICS IN PROBABILITY. PART II: UNIVERSALITY

Exercise sheet 8: Universality and strictly stable distributions

Exercise 1 (Characteristic functions of strictly stable laws)**.**

Let $\phi_{\alpha,a}(t) = \exp(-a|t|^\alpha)$. Show that $\phi_{\alpha,a}$ is a characteristic function of a random variable iff $a > 0$, $\alpha \in (0, 2]$. Show that the latter ones are exactly characteristic functions of symmetric strictly stable α distributions.

Remark: Note that there are strictly stable distributions whose characteristic exponent is more complicated and involves imaginary terms, e.g. standard Lévy distribution or generalized Cauchy distribution (with non-zero location parameter).

Proof. Part 1: Let us first notice that α has to be positive since otherwise ϕ would have a singularity at zero or would be just a constant function e^{-a}, both can not happen if ϕ is a characteristic function. The same argument implies that a can not be 0 (except for the case $X = 0$, which we exclude). Moreover, if a were negative, we would have $\phi(t) = e^{c|t|^\alpha}$ for a positive constant c, which is an unbounded function. This can not happen since the absolute value of a characteristic function is bounded by 1. So, we are left with the range $a > 0, \alpha > 0$.

Let us show that for $\alpha > 2$, $\phi_{\alpha,a}$ can not be a characteristic function. To this end, note that if X has characteristic function $\phi_{\alpha,a}$, then its law is symmetric and α-stable[1] with norming constants $c_n = n^{1/\alpha}$ (check it). Since the only strictly stable distribution with finite variance is normal (corresponds to $\alpha = 2$) — this follows by CLT — it suffices to show that if $\alpha > 2$, then the variance of X is finite. Let t be sufficiently large such that $\mathbb{P}[|X| > t] < 1/4$. Then also $\mathbb{P}[|S_n| > c_n t] < 1/4$ for $S_n = \sum_{i=1}^n X_i$. By symmetry of the distribution, we further obtain that $\mathbb{P}[|S_n| > c_n t] \geq \frac{1}{2}\mathbb{P}[\max_{i\leq n} |X_i| > t]$. Indeed, let X^1 be the first term among $X_1, \dots, X_n$ that is largest in the absolute value and $S_n' := S_n - X^1$; then by i.i.d. assumption on X_i's and symmetry of their law, the joint laws of (X^1, S_n') and $(X^1, -S_n')$ coincide. Thus,

$$\mathbb{P}[X^1 > c] \leq \mathbb{P}[X^1 > 0, S_n' \leq 0] + \mathbb{P}[X^1 > 0, S_n' \geq 0] = 2\mathbb{P}[X^1 > 0, S_n' \geq 0],$$

which, in turn, implies that

$$\mathbb{P}[|S_n| > c_n t] = 2\mathbb{P}[S_n > c_n t] \geq 2\mathbb{P}[X^1 > c_n t, S_n' \geq 0] \geq \mathbb{P}[X^1 > c_n t] = \frac{1}{2}\mathbb{P}[\max_{i\leq n} |X_i| > c_n t].$$

by symmetry of X^1, as desired. We furthermore have

$$\mathbb{P}[\max_{i\leq n} |X_i| > t] = 1 - (1 - \mathbb{P}[|X| > c_n t])^n = 1 - e^{n \log(1-\mathbb{P}[|X|>c_n t])} \geq 1 - e^{-n\mathbb{P}[|X|>c_n t]}.$$

Altogether, we have shown that

$$\frac{1}{2}(1 - \exp(-n\mathbb{P}[|X| > tc_n])) \leq \mathbb{P}[|Y_n| > tc_n] < \frac{1}{4}.$$

[1] X has strictly α-stable law if for any n there exists $c_n \in \mathbb{R}$ such that $\frac{1}{c_n}\sum_{i=1}^n X_i$ for i.i.d. copies X_i of X has the same law as X

This implies that $(n\mathbb{P}[|X| > tc_n])_n$ is bounded, and since $c_n = n^{1/\alpha}$, also that $\mathbb{P}[|X| > x] \leq Mx^{-\alpha}$ for some $M > 0$ and all x sufficiently large. This tail-asymptotic immediately yields that $\mathrm{Var}[X]$ is finite iff $\alpha > 2$.

So far we have shown that outside of the range $a > 0, \alpha \in (0, 2]$, $\phi_{\alpha,a}$ is not a characteristic function. Note that wlog we might restrict ourselves to $a = 1$, since for any other positive value b we obtain the desired characteristic function by considering $b^\alpha X$ for X with $\phi_{\alpha,1}$. For $\alpha = 2$, $\phi_{2,a}$ is the characteristic function of centered normal distribution (if $a = 1$, then with variance $\sqrt{2}$). So, it remains to show (for the first statement) that $\phi_{\alpha,1}$ is a characteristic function of some distribution for $\alpha \in (0, 2)$. This follows immediately from Exercise 3.

<u>Part 2:</u> We now complete the proof of the exercise by verifying that characteristic functions of symmetric strictly α-stable distributions (for $\alpha \in (0, 2)$, $\alpha = 2$ corresponds to Gaussian) are of the form $\phi_{\alpha,a}$ for some $a > 0$. To this end, we show that the norming constants are given by $c_n = n^{1/\alpha}$, which then together with stability yields that for any $m, n \in \mathbb{N}$, $\phi_X(m/n) = \phi_X(1)^{|m|^\alpha/|n|^\alpha}$, and thus, $\phi_X(q) = \phi(1)^{|q|^\alpha}$ for all rational q. Using this relation and continuity of characteristic functions, we may conclude that $\phi_X(u) > 0$ for all u, so $c = -\log \phi(1) > 0$ and $\phi_X(t) = e^{-c|t|^\alpha}$ for all $t \in \mathbb{R}$.

For the norming constants, by stability we obtain,

$$c_{nm} X \overset{\text{law}}{=} Y_{mn} \overset{\text{law}}{=} \sum_{k=1}^{m} \sum_{i=(k-1)n+1}^{kn} X_i \overset{\text{law}}{=} \sum_{k=1}^{m} c_n X_k \overset{\text{law}}{=} c_m c_n X;$$

which by induction, in turn, leads to $c_{n^k} = c_n^k$ for all $n, k \in \mathbb{N}$. Analogously, by stability we conclude that

$$X \overset{\text{law}}{=} \frac{1}{c_{n+m}} Y_{m+n} \overset{\text{law}}{=} \frac{c_m}{c_{n+m}} X_1 + \frac{c_n}{c_{n+m}} X_2 := U.$$

Let $x > 0$, then by symmetry of distribution of X and independence:

$$\mathbb{P}[X > x] = \mathbb{P}[U > x] \geq \mathbb{P}\left[X_1 \geq 0, X_2 > \frac{c_{m+n}}{c_n}\right] \geq \frac{1}{2}\mathbb{P}\left[X > \frac{c_{m+n}}{c_n}\right].$$

This yields that c_n/c_{n+m}'s are bounded uniformly for all $m, n \in \mathbb{N}$ since otherwise there would have been a sequence of m, n such that $c_{n+m}/c_n \to 0$, which would lead to $\mathbb{P}[X > x] \geq 1/4$ for all $x > 0$. Contradiction, as X is a well-defined real-valued random variable. In particular, c_k/c_n's are bounded for all $k < n$.

Fix $r, s \in \mathbb{N}$, then there are unique $\alpha, \beta \in (0, \infty)$ such that $c_r = r^{1/\alpha}$, $c_s = s^{1/\beta}$. Now, by the above observations we conclude that $c_n = n^{1/\alpha}$ for all $n = r^k$ with $k \in \mathbb{N}$ and $c_m = m^{1/\beta}$ for all $m = s^l$ with $l \in \mathbb{N}$. The goal is to show that $\alpha = \beta$, then it will follow that $c_n = n^{1/\alpha}$ for all $n \in \mathbb{N}$. Towards that end, note that for any $m = s^l$, there exists $n = r^k$ so that $n < m \leq rn$. Hence,

$$c_m = m^{1/\beta} \leq (nr)^{1/\beta} = r^{1/\beta} c_n^{\alpha/\beta}.$$

Since c_n/c_m's are bounded for all $m > n$ as above and since c_m's are unbounded in m (follows from definition of stability), we obtain that $\alpha \geq \beta$. Reversing the roles of n and m gives that $\beta \leq \alpha$, so $\alpha = \beta$. $\square$

Exercise 2 (Strictly stable distribution: equivalent definition).

1) Random variable X has a strictly stable distribution iff the following condition holds: if X_1, X_2 are independent variables, each with the same distribution as X and $a, b > 0$, then there exists a constant $c > 0$ such that $aX_1 + bX_2$ has the same distribution as cX.

Hint: for "only if" direction, find the explicit norming parameters using previous exercise, understand for which choices of a, b one could directly conclude and how to extend the result to all $a, b > 0$.

2) Prove that if X has a strictly stable distribution with norming constant c_n, then so is the law of $-X$. Conclude that if Y is independent of X with the same distribution, then the distribution of $X - Y$ is strictly stable with the same norming constant. Note that law of $X - Y$ is as well symmetric w.r.t. 0.

Proof. 1) Suppose that the condition holds, by induction we will conclude that the definition holds: for $n = 2$ follows by taking $a = b = 1$. Suppose we know that there exist norming parameters $c_1, \dots, c_n > 0$ such that $\frac{1}{c_k} \sum_{i \le n} X_i$ has the same law as X for $k \le n$. Here X_i independent copies of X. By independence $X_1 + \ldots + X_{n+1}$ has the same distribution as $c_n X + X_{n+1}$. The latter, in turn, by our condition has the same distribution as $c\tilde{X}$ for some $c > 0$, $\tilde{X}$ is an independent copy of X. Thus, by setting $c_{n+1} = c$, the claim follows.

For the reverse direction, we would use the fact that norming constants of X are given by $c_n = n^{1/\alpha}$ for some $\alpha > 0$ (which follows immediately from Exercise 1). This together with stability implies that for any $m, n \in \mathbb{N}$, $n^{1/\alpha} X_1 + m^{1/\alpha} X_2$ has the same distribution as $\sum_{i \le m+n} X_i$ which, in turn, has the same law as $(n+m)^{1/\alpha} X$. By dividing both sides by an appropriate $k^{1/\alpha}$ for $k \in \mathbb{N}$, we obtain that for all rational $q, p > 0$, there exists $c > 0$ such that $q^{1/\alpha} X_1 + p^{1/\alpha} X_2 \overset{\text{law}}{=} cX$. Note that it suffices to work with the case $a = 1$ and $b = r^{1/\alpha}$ for any $r > 0$. Recall that we have the claim for $b = q^{1/\alpha}$ for an arbitrary rational $q > 0$. But now by continuity of the stable distribution (follows from Exercise 1, the degenerate case is trivial), we conclude that taking the limit $q \in \mathbb{Q} \to r$ will deliver the result, the coefficient $c(r) = \lim_q c(q)$.

2) Let us show that $-X$ has stable law with the same norming parameters c_n as X. Let $X_1, \dots, X_n$ be independent copies of X, then $-X_1, \dots, -X_n$ is a sequence of independent variables each with the same distribution as $-X$. By stability $\sum_{i \le n} -X_i = -\sum_{i \le n} X_i$ has law of $c_n(-X)$. Let X, Y be independent with the same distribution, let $Z_1, \dots, Z_n$ be a sequence of independent variables with the same distribution as $X - Y$, then $\sum_i Z_i$ has the same distribution as $\sum_i X_i - \sum_i Y_i$, where X_i are independent variables with law of X, and Y_i are independent with law Y, so that all X_i, Y_j are mutually independent. By stability, $\sum_i X_i - \sum_i Y_i$ has the same distribution as $c_n X - c_n Y = c_n(X - Y)$. $\square$

Exercise 3 (Heavy tailed CLT)**.**

Consider a random variable X, whose law is symmetric around zero and that satisfies $\mathbb{P}[|X| > x] = x^{-\alpha}$ for $\alpha \in (0, 2)$ for all $x \ge 1$. Now let $X_1, X_2, \ldots$ be i.i.d. with the law of X. E.g. by using characteristic functions (or otherwise) prove that $\frac{1}{n^{1/\alpha}} \sum_{i=1}^n X_i$ converges to the symmetric α strictly stable law, i.e. the norming constant is $c_n = n^{1/\alpha}$ and the law is symmetric w.r.t. zero.

Proof. Note that from $\mathbb{P}[|X| > x] = x^{-\alpha}$ for $\alpha \in (0, 2)$ for all $x \ge 1$, we conclude that the random variable is supported on $\mathbb{R} \setminus [-1, 1]$ and its density is given by $f_X(x) = \frac{\alpha/2}{|x|^{\alpha+1}} \mathbf{1}_{|x| \ge 1}$ (check it). Hence, the characteristic function of X, $\phi_X(t) = \int_{\mathbb{R}} e^{itx} f_X(x) \mathrm{d}x = \int_1^\infty \cos(|t|x) \frac{\alpha}{x^{\alpha+1}} \mathrm{d}x$.

We have that for any $t \neq 0$ (interested in $|t|$ small)

$$\phi_X(t) = 1 - \int_1^\infty (1 - \cos(|t|x)) \frac{\alpha}{x^{\alpha+1}} \mathrm{d}x = |t|^\alpha \underbrace{\int_{|t|}^\infty (1 - \cos u) \frac{\alpha}{u^{\alpha+1}} \mathrm{d}u}_{\int_{|t|}^1 + \int_1^\infty}.$$

Note that for $|t|$ small, in the above splitting the second integral is positive (as integrand is almost everywhere positive) and finite, the value does not depend on $|t|$. As for the first integral, note that $1 - \cos u \approx u^2/2$ for small values of u, thus, the first integral is bounded by $c(\alpha) + b(\alpha)|t|^{2-\alpha}$, where $c(\alpha)$ is the constant appearing by evaluating F at $x = 1$ and $b(\alpha) > 0$ is an explicit constant depending only on α. Therefore, as $|t|$ tends to 0, $\phi_X(t) \sim 1 - C|t|^\alpha(1 + o(1))$. Thus, as $n \to \infty$

$$\phi_{\frac{\sum^n X_i}{n^{1/\alpha}}}(t) = \phi_X(t/n^{1/\alpha})^n \sim \left(1 - C\frac{|t|^\alpha(1 + o(1))}{n}\right)^n \sim e^{-C|t|^\alpha}.$$

Those are exactly the characteristic functions of α stable symmetric distributions - see Exercise 1. □

Exercise 4 ("Open question")**.** *Find classes of functions $f_n : \mathbb{R}^n \to \mathbb{R}$ such that $f_n \in C^\infty$ and so that*

$$\sup_x \left| \frac{\partial^3}{\partial x_i^3} f_n(x) \right| = o(n^{-1}) \quad \forall 1 \leq i \leq n,$$

for all n sufficiently large.

Proof. Proof of this exercise will be provided in a separate file. □