TOPICS IN PROBABILITY. PART III: PHASE TRANSITION

Exercise sheet 11: Sharp phase transition and Hypercontractivity

Exercise 1 (Russo's almost $0-1$ law)**.** *Consider a product measure $\mathbb{P}_p$ on $\{0,1\}^n$: each 1 is assigned weight p. Prove the following result: for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any increasing event A with the associated ith influence satisfying $I_i^p(A)(= \mathbb{P}_p[\mathbf{1}_A(\bar{X}) \neq \mathbf{1}_A(\bar{X}^{(i)}]) \leq \delta$ for all i, p, there exists p_c such that for all $p \geq p_c + \varepsilon$, $\mathbb{P}_p[A] \geq 1 - \varepsilon$, and for all $p \leq p_c - \varepsilon$, $\mathbb{P}_p[A] \leq \varepsilon$.*

Proof. Let $\varepsilon > 0$, and $\delta = \delta(\varepsilon) > 0$ (to be specified later). Let A be an arbitrary increasing event such that $I_i^p(A) \leq \delta$ for all i, p. We choose p_c (depending on A) such that $\mathbb{P}_{p_c}[A] = 1/2$. Note that the latter p_c indeed exists and is in $(0,1)$ since $p \mapsto \mathbb{P}_p[A]$ is increasing with $\{\mathbb{P}_0[A], \mathbb{P}_1[A]\} = \{0,1\}$. Consider $\varepsilon_0 := \min(1-p_c, p_c)/2$. By monotonicity, it suffices to prove that $\mathbb{P}_{p_c+\varepsilon\wedge\varepsilon_0}[A] \geq 1 - \varepsilon \wedge \varepsilon_0$ and $\mathbb{P}_{p_c-\varepsilon\wedge\varepsilon_0}[A] \leq \varepsilon \wedge \varepsilon_0$. By Talagrand's inequality, for an appropriate absolute constant $c > 0$,

$$
\begin{aligned}
\mathrm{Var}_p[\mathbf{1}_A] &\leq c \log\Big(\frac{1}{p(1-p)}\Big) \sum_{i=1}^n \frac{I_i^p(A)}{\log(1/I_i^p(A))} \\
&\leq c \log\Big(\frac{1}{p(1-p)}\Big) \frac{1}{\log(1/\delta)} \sum_{i=1}^n I_i^p(A) \leq c(p_c) \frac{1}{\log(1/\delta)} \sum_{i=1}^n I_i^p(A)
\end{aligned}
$$

for all $p \in [p_c - \varepsilon_0, p_c + \varepsilon_0]$. By Lemma 2.3,

$$
\begin{aligned}
&\mathbb{P}_{p_c+\varepsilon\wedge\varepsilon_0}[A] \geq 1 - \exp\Big(-\log(1/\delta)\frac{\varepsilon\wedge\varepsilon_0}{c(p_c)}\Big) = 1 - \delta^{\frac{\varepsilon\wedge\varepsilon_0}{c(p_c)}}; \\
&\mathbb{P}_{p_c-\varepsilon\wedge\varepsilon_0}[A] \leq \delta^{\frac{\varepsilon\wedge\varepsilon_0}{c(p_c)}}.
\end{aligned}
$$

Choose $\delta = (\varepsilon \wedge \varepsilon_0)^{c(p_c)/\varepsilon\wedge\varepsilon_0}$, then as desired for all $p \geq p_c + \varepsilon$, $\mathbb{P}_p[A] \geq 1 - \varepsilon \wedge \varepsilon_0$ and for all $p \leq p_c - \varepsilon$, $\mathbb{P}_p[A] \leq \varepsilon \wedge \varepsilon_0$. $\square$

Exercise 2 (Heat semigroup on the hypercube)**.**

Let $\mu = \mu_1 \otimes \ldots \otimes \mu_n := \big(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1\big)^{\otimes n}$ be the uniform measure on the hypercube $\{-1,1\}^n$. Define the process $X_t = (X_t^1, \ldots, X_t^n)_{t\geq 0}$ as follows. To each coordinate i, we attach an independent Poisson process $(N_t^i)_{t\geq 0}$ of intensity one[1]. Then,

- *draw $X_0 \sim \mu$ independently from the PP $N = (N^1, \ldots, N^n)$;*
- *each time N_t^i jumps for some i, replace the current value of X_t^i by an independent sample from μ_i while keeping the remaining coordinates fixed.*

We define $P_t f(x) := \mathbb{E}[f(X_t) | X_0 = x]$. Prove the following properties of $(X_t)_t$ and its associated semigroup $(P_t)_t$:

[1] Poisson process $(N_t)_{t\geq 0}$ of intensity $\lambda > 0$ on $\mathbb{R}_+$ is the counting process that satisfies $N_0 = 0$, has independent increments and $N_t - N_s \sim \mathrm{Poisson}(\lambda(t-s))$ for any $t \geq s \geq 0$. Recall that such a process can be constructed, for instance, in the following way: let $(T_i)_i$ be i.i.d. exponential random variables of intensity λ, then $N_t := \#\{n \in \mathbb{N}_0 : \sum_{i=1}^n T_i \leq t\}$ is the Poisson process of intensity λ.

(1) $(X_t)_t$ *satisfies the Markov property;*
(2) μ *is its stationary measure, i.e., that* $\int \mathbb{E}[f(X_t)|X_0 = x]\mu(\mathrm{d}x) = \int f\mu(\mathrm{d}x)$*;*
(3) $P_t f(x) = \sum_{S\subset\{1,\dots,n\}}(1-e^{-t})^{|S|}e^{-t(n-|S|)}\int f(x_1,\dots,x_n)\prod_{i\in S}\mu_i(\mathrm{d}x_i)$*;*
(4) *Each* $f : \{-1,1\}^n \to \mathbb{R}$ *can be written as* $f = \sum_{S\subset\{1,\dots,n\}}\hat{f}(S)u_S$ *for appropriate coefficients* $\hat{f}(S)$ *and* $u_S(x) = \prod_{i\in S} x_i$[2]. *Recall that using this representation, you defined in class an operator* $T_{e^{-t}}f(x) = \sum_{S\subset\{1,\dots,n\}} e^{-t|S|}\hat{f}(S)u_S$ *for any* $t \geq 0$. *Show that* $T_{e^{-t}} = P_t$.
(5)

$$\mathcal{L}f := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0+} P_t f(x) := \lim_{t\downarrow 0}\frac{P_t f - f}{t} = -\sum_{i=1}^n \delta_i f,$$

where $\delta_i f(x) = f(x) - \int f(x_1,\dots,y_i,\dots,x_n)\mu_i(\mathrm{d}y_i) = f(x) - \frac{f(x_1,\dots,1,\dots,x_n)+f(x_1,\dots,-1,\dots,x_n)}{2}$. *Note that* $\delta_i\delta_i = \delta_i$ *and* $\delta_i\delta_j = \delta_j\delta_i$*, hence,* $-\sum_i \delta_i$ *is a discrete Laplacian. Thus, let us write* Δ *instead of* $\mathcal{L}$.
(6) *Conclude that* $\frac{\mathrm{d}}{\mathrm{d}t}P_t f = \Delta P_t f$*;*
(7) $\delta_i P_t f = P_t \delta_i f$.

Proof. Let $(X^i_m)_{i\leq n, m\in\mathbb{N}}$ be iid Rademacher variables independent of N. Then $(X_t)_t$ has the same law as $(X^1_{N^1_t},\dots,X^n_{N^n_t})_t$. Using this observation, the Markov property follows directly from the independence of increments of the Poisson process, and stationarity of μ from the independence of X^i_m's and N and identical distribution of $(X^i_m)_m$'s (use tower property). Let us determine the transition semigroup and the generator of $(X_t)_t$,

$$\begin{aligned}
P_t f(x) = \mathbb{E}[f(X_t)|X_0 = x] &= \sum_{S\subset\{1,\dots,n\}} \mathbb{E}_x[f(X_t)(\mathbf{1}_{N^i_t>0 \text{ for } i\in S}\mathbf{1}_{N^i_t=0 \text{ for } i\notin S})] \\
&= \sum_{S\subset\{1,\dots,n\}}(1-e^{-t})^{|S|}e^{-t(n-|S|)}\int f(x_1,\dots,x_n)\prod_{i\in S}\mu_i(\mathrm{d}x_i) \\
&= \sum_{S\subset\{1,\dots,n\}}\sum_{S'\subset\{1,\dots,n\}}\hat{f}(S')(1-e^{-t})^{|S|}e^{-t(n-|S|)}\int\prod_{i\in S'}x_i\prod_{i\in S}\mu_i(\mathrm{d}x_i) \\
&= \sum_{S\subset\{1,\dots,n\}}\sum_{S'\subset\{1,\dots,n\}}\hat{f}(S')(1-e^{-t})^{|S|}e^{-t(n-|S|)}\prod_{i\in S'\setminus S}x_i\underbrace{\prod_{j\in S\cap S'}\int x_j\mu_j(\mathrm{d}x_j)}_{=\mathbf{1}_{\{S\cap S'=\emptyset\}}} \\
&= \sum_{S'\subset\{1,\dots,n\}}\hat{f}(S')\prod_{i\in S'}x_i\underbrace{\sum_{S\subset\{1,\dots,n\}\setminus S'}(1-e^{-t})^{|S|}e^{-t(n-|S|)}}_{=e^{-t|S'|}\sum_{k=0}^{n-|S'|}\binom{n-|S'|}{k}(1-e^{-t})^k(e^{-t})^{n-|S'|-k}=e^{-t|S'|}} \\
&= \sum_{S'\subset\{1,\dots,n\}}e^{-t|S'|}\hat{f}(S')u_{S'} = T_{e^{-t}}f;
\end{aligned}$$

$$\mathcal{L}f(x) = -nf(x) + \sum_{k=1}^n\int f(x_1,\dots,x_n)\mu_k(\mathrm{d}x_k) = -\sum_{k=1}^n\delta_i f(x).$$

[2]this definition of u_S might differ from the one used in class by a factor of $(-1)^{|S|}$

By Markov property (and time homogeneity), we further get that $P_{t+s}f(x) = \mathbb{E}[\mathbb{E}[f(X_{t+s})|(X_u)_{u\leq t}]|X_0 = x] = \mathbb{E}[\tilde{\mathbb{E}}[f(\tilde{X}_s)|\tilde{X}_0 = X_t]|X_0 = x] = \mathbb{E}[P_s f(X_t)|X_0 = x] = P_t(P_s f)(x)$. Thus, $\frac{\mathrm{d}}{\mathrm{d}t}P_t f = \lim_{u\downarrow 0} \frac{P_{t+u}f - P_t f}{u} = lim_{u\downarrow 0}\frac{P_u(P_t f) - P_t f}{u} = \mathcal{L}(P_t f) = \Delta P_t f$. Since $P_0 = \mathrm{Id}$, this in turn implies that $P_t = e^{t\Delta}$, but since all δ_j's commute with one another, it follows that $P_t = \prod_{j=1}^n e^{-t\delta_j}$. Clearly, P_t and δ_i commute. [3] □

⋆ For fun.

Exercise 3 (Hypercontractivity and log-Sobolev on the hypercube)**.** *Let P_t, μ be as in the previous exercise. Prove that the following are equivalent:*

(1) (log-Sobolev) $\mathrm{Ent}_\mu[f^2] \leq \frac{1}{2}\mathbb{E}_\mu[\sum_{i=1}^n (2\delta_i f)^2]$

(2) (Hypercontractivity) $\|P_t f\|_{\mathrm{L}^{q(t)}(\mu)} \leq \|f\|_{\mathrm{L}^q(\mu)}$ *for all* $q \geq 1, t \geq 0,$ $f \in \mathrm{L}^q(\mu)$ *and* $q(t) = 1 + (q-1)e^{2t}$.

To show the implication (1) ⇒ (2) *you may proceed as follows:*

- *Reduce to the case* $f \geq 0$.
- *Consider* $\log \|P_t f\|_{\mathrm{L}^{q(t)}}$ *and compute its derivative.*
- *Show that the derivative is non-positive by reducing to the following problem: for a function* $g \geq 0$, $p = q(t)$ *and* $p' = q'(t)$,

$$\sum_{i=1}^n \left(\mathbb{E}\left[\left(g^{p/2} - g_{(i)}^{p/2} \right)^2 \right] - \frac{p^2}{p'}\mathbb{E}[g^{p-1}\delta_i g] \right) \leq 0. \tag{0.1}$$

  Hint: apply log-Sobolev in this step.
- *Prove* (0.2) *using* $\left(\frac{b^{p/2}-a^{p/2}}{b-a}\right)^2 \leq \frac{p^2}{4(p-1)}\frac{b^{p-1}-a^{p-1}}{b-a}$ *(check this) and the previous exercise.*

To show the converse implication you may want to re-use the above second step.

Proof. Let $q \geq 1$ and $q(t) = 1 + (q-1)e^{2t}$. Note that $|P_t f| \leq P_t|f|$, therefore, we can restrict to the case $f \geq 0$. Note that $\log \|P_t f\|_{\mathrm{L}^{q(t)}} = \frac{1}{q(t)} \log \mathbb{E}[(P_t f)^{q(t)}]$ and so,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{q(t)} \log \mathbb{E}[(P_t f)^{q(t)}] \right) \\
&\quad = \frac{-q'(t)}{q^2(t)} \log \mathbb{E}[(P_t f)^{q(t)}] + \frac{1}{q(t)}\frac{1}{\mathbb{E}[(P_t f)^{q(t)}]}\mathbb{E}\left[(P_t f)^{q(t)} \left(q'(t)\log(P_t f) + q(t)\frac{1}{P_t f}\frac{\mathrm{d}}{\mathrm{d}t}(P_t f) \right) \right] \\
&\quad = \frac{q'(t)}{q(t)^2 \mathbb{E}[(P_t f)^{q(t)}]} \left(\mathrm{Ent}[(P_t f)^{q(t)}] + \frac{q(t)^2}{q'(t)}\mathbb{E}[(P_t f)^{q(t)-1}\mathcal{L}(P_t f)] \right) \\
&\quad \overset{\text{log-Sobolev}}{\leq} \frac{q'(t)}{q(t)^2 \mathbb{E}[(P_t f)^{q(t)}]} \sum_{i=1}^n \left(\mathbb{E}\left[\left((P_t f)^{q(t)/2} - (P_t f)_{(i)}^{q(t)/2} \right)^2 \right] - \frac{q(t)^2}{q'(t)}\mathbb{E}[(P_t f)^{q(t)-1}\delta_i(P_t f)] \right),
\end{aligned}$$

where $f_{(i)}(Y) = f(Y^1, \dots, Y^{i-1}, \tilde{Y}^i, Y^{i+1}, \dots, Y^n)$, where $\tilde{Y}_i$ is an independent copy of Y_i. Note that the coefficient in front of the sum is positive. Thus, to prove that the derivative

[3] Note that to prove the latter two bullet-points of the exercise, one may also proceed differently and use an explicit form obtained in (4).

is non-positive, it suffices to show that for $g \geq 0$, $p = q(t)$ and $p' = q'(t)$,

$$\sum_{i=1}^{n} \left(\mathbb{E}\left[\left(g^{p/2} - g_{(i)}^{p/2} \right)^2 \right] - \frac{p^2}{p'} \mathbb{E}[g^{p-1} \delta_i g] \right) \leq 0. \tag{0.2}$$

Note that for $a, b \geq 0$, by Cauchy-Schwarz, $(b^{p/2} - a^{p/2})^2 = \left(\frac{p}{2} \int_a^b u^{p/2-1} \mathrm{d}u \right)^2 \leq \frac{p^2(b-a)}{4} \int_a^b u^{p-2} = \frac{p^2(b-a)(b^{p-1}-a^{p-1})}{4(p-1)}$. Hence, by (0.2) it suffices to show that

$$\begin{aligned}\sum_{i=1}^{n} \frac{p^2}{2(p-1)} &\left(\underbrace{\frac{1}{2} \mathbb{E}\left[(g - g_{(i)})(g^{p-1} - g_{(i)}^{p-1}) \right]}_{= \mathbb{E}\left[g(g^{p-1} - g_{(i)}^{p-1}) \right]} - \frac{2(p-1)}{p'} \underbrace{\mathbb{E}[g^{p-1} \delta_i g]}_{= \mathbb{E}\left[g(g^{p-1} - g_{(i)}^{p-1}) \right]} \right) \\ &= \sum_{i=1}^{n} \frac{p^2}{2(p-1)} \mathbb{E}\left[g(g^{p-1} - g_{(i)}^{p-1}) \right] \left(1 - \frac{2(p-1)}{p'} \right) \leq 0.\end{aligned}$$

Note that $1 - 2(p-1)/p' = 1 - 2(q-1)e^{2t}/(2e^{2t}(q-1)) = 0$. All together, we have shown that $\log \|P_t f\|_{\mathrm{L}^{q(t)}}$ is non-increasing; $q(0) = q$. Therefore, $\|P_t f\|_{\mathrm{L}^{q(t)}} \leq \|f\|_{\mathrm{L}^q}$ as desired.

For the reverse implication, note that by (2), $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \log \|P_t f\|_{\mathrm{L}^{q(t)}} \leq 0$. And hence by the above and the previous exercise with $q = 2$ (hence, $q(0) = 2, q'(0) = 2$), we get that

$$0 \geq \frac{1}{2\mathbb{E}[f^2]} \left(\mathrm{Ent}[f^2] + 2\mathbb{E}[f \mathcal{L} f] \right) = \frac{1}{2\mathbb{E}[f^2]} \left(\mathrm{Ent}[f^2] - 2 \sum_i \mathbb{E}[f \delta_i f] \right).$$

We may yield the desired result since $2\mathbb{E}[f \delta_i f] = \mathbb{E}[f \delta_i f] - \mathbb{E}[f^i \delta_i f] = 2\mathbb{E}[(\delta_i f)^2]$, where f^i stands for f with flipped ith coordinate. $\square$