TOPICS IN PROBABILITY. PART III: PHASE TRANSITION

Exercise sheet 10: Monotonicity and sharp transitions

Exercise 1 (Why monotonicity assumption?)**.**

Recall that from Margulis-Russo lemma you have concluded that $p \mapsto \mathbb{E}_p[f]$ is nondecreasing whenever f is monotone, i.e. $f(x) \le f(y)$ whenever $x_i \le y_i$ for all i. Conclude this result easier in the following way:

- *Let $p \le p'$. Show that there exists a coupling (X, X') of $\mathbb{P}_p$ and $\mathbb{P}_{p'}$ such that $X_i \le X'_i$ a.s. for every $i = 1, \dots, n$. Conclude that $p \mapsto \mathbb{E}_p[f]$ is non-decreasing whenever f is monotone.*

When f is not monotone, the map $p \mapsto \mathbb{E}_p[f]$ can be essentially arbitrary, as the following exercise shows:

- *Given any continuous function $h : [0,1] \to [0,1]$, construct functions $f_n : \{0,1\}^n \to \{0,1\}$ so that $\mathbb{E}_p[f_n] \to h(p)$ as $n \to \infty$ for all $p \in (0,1)$.*

For this reason, there is no meaningful notion of a (sharp) transition in general, unless an assumption such as monotonicity is made.

Proof. Let us first prove the monotonicity of $p \mapsto \mathbb{E}_p[f]$: Let $(U_i)_i$ be a family of independent random variables that are uniformly distributed over $[0,1]$. Set $X = (X_i)_i := (\mathbf{1}_{U_i \le p})$ and $X' = (X'_i)_i := (\mathbf{1}_{U_i \le p'})$. Then clearly $X \sim \mathbb{P}_p$ and $X' \sim \mathbb{P}_{p'}$. Moreover, $\mathbb{P}[X \le X'] = \mathbb{P}[\mathbf{1}_{U_i \le p} \le \mathbf{1}_{U_i \le p'} \ \forall i] = 1$. In particular, for any monotone function f, $\mathbb{E}_p[f] = \mathbb{E}[f(X)] \le \mathbb{E}[f(X')] = \mathbb{E}_{p'}[f]$ as desired.

Let $n \in \mathbb{N}$ be fixed, define E_i^n to be the subset of $\{0,1\}^n$ which consists of elements with exactly i coordinates being 1. Note that $\{0,1\}^n = \dot{\bigcup}_i E_i$. Order the elements of E_i^n in lexicographical order (e.g., $001 \le 010 \le 100$ in E_1^3). Denote the $\lfloor h(i/n)\binom{n}{i} \rfloor$'s element in this chain by U_i^n. Then f_n on each E_i^n is given by $\mathbf{1}_{(x_j)_j \le U_i^n}$. Using this and Stirling's formula, we get the following

$$\begin{aligned}
\mathbb{E}_p[f_n] &= \sum_{i=0}^{n} \mathbb{E}_p[f_n \mathbf{1}_{E_i^n}] = \sum_{i=0}^{n} \sum_{(x_j)_j \in E_i^n} f_n((x_j)_j) p^i (1-p)^{n-i} \\
&= \sum_i \left\lfloor h(i/n) \binom{n}{i} \right\rfloor p^i (1-p)^{n-i} \\
&\overset{n\to\infty}{\sim} \sum_{i=1}^{n-1} \frac{h(i/n)}{\sqrt{2\pi n}} \frac{1}{\sqrt{(i/n)(1-i/n)}} \left(\left(\frac{pn}{i}\right)^{i/n} \left(\frac{1-p}{1-i/n}\right)^{1-i/n} \right)^n \\
&\overset{n\to\infty}{\sim} \int_{1/n}^{1} h(x) \sqrt{\frac{n}{2\pi}} \frac{1}{\sqrt{x(1-x)}} \left(\left(\frac{p}{x}\right)^x \left(\frac{1-p}{1-x}\right)^{1-x} \right)^n \mathrm{d}x.
\end{aligned}$$

Note that $x \mapsto \left(\frac{p}{x}\right)^x \left(\frac{1-p}{1-x}\right)^{1-x}$ is strictly increasing on $(0,p)$ and strictly decreasing on $(p,1)$. It attains its maximum 1 at $x = p$. Therefore, since outside of $(p-\varepsilon, p+\varepsilon)$ (ε to be determined, it tends to 0 as $n \to \infty$) the above integrand decays uniformly exponentially fast, we get

$$
\begin{aligned}
\mathbb{E}_p[f_n] &\overset{n\to\infty}{\sim} \int_{p-\varepsilon}^{p+\varepsilon} h(x)\sqrt{\frac{n}{2\pi}}\frac{1}{\sqrt{x(1-x)}}\left(\left(\frac{p}{x}\right)^x\left(\frac{1-p}{1-x}\right)^{1-x}\right)^n \mathrm{d}x \\
&\overset{n\to\infty}{\sim} \int_{p-\varepsilon}^{p+\varepsilon} h(x)\sqrt{\frac{n}{2\pi}}\frac{1}{\sqrt{x(1-x)}}\left(1-\frac{(x-p)^2}{2p(1-p)}\right)^n \mathrm{d}x \\
&\approx h(p)\frac{1}{\sqrt{p(1-p)}}\sqrt{\frac{n}{2\pi}}\int_{-\varepsilon}^{+\varepsilon} \underbrace{\left(1-\frac{x^2}{2p(1-p)}\right)^n}_{\approx \exp(-x^2n/(2p(1-p)))} \mathrm{d}x \\
&\overset{n\to\infty}{\sim} h(p)\sqrt{\frac{n}{2\pi}}\frac{1}{\sqrt{p(1-p)}}\int_{-\varepsilon\sqrt{n}/\sqrt{p(1-p)}}^{+\varepsilon\sqrt{n}/\sqrt{p(1-p)}} \frac{\sqrt{p(1-p)}}{\sqrt{n}}\exp\left(-y^2/2\right)\mathrm{d}x \\
&\overset{n\to\infty}{\sim} h(p)\frac{1}{\sqrt{2\pi}}\int_{-\varepsilon\sqrt{n}/\sqrt{p(1-p)}}^{+\varepsilon\sqrt{n}/\sqrt{p(1-p)}} \exp\left(-y^2/2\right)\mathrm{d}x \overset{n\to\infty}{\sim} h(p),
\end{aligned}
$$

where in the second step we expanded $\left(\frac{p}{x}\right)^x\left(\frac{1-p}{1-x}\right)^{1-x}$ at $x=p$ and in the last line we have used that $\varepsilon\sqrt{n}$ tends to infinity. Indeed, to determine $\varepsilon>0$ such that $n^{1/2n}\left(\frac{p}{x}\right)^x\left(\frac{1-p}{1-x}\right)^{1-x}<1$ uniformly outside of $(p-\varepsilon,p+\varepsilon)$, expand $\left(\frac{p}{x}\right)^x\left(\frac{1-p}{1-x}\right)^{1-x}$ at $x=p$ and $n^{1/2n}$ at ∞ and compare the coefficients. This results into $\varepsilon^2>p(1-p)\log n/n$, so take $\varepsilon=2\sqrt{p(1-p)\log n/n}$. The proof is, thus, complete. □

Exercise 2 (Examples of Boolean functions: influence and presence or absence of a phase transition.)**.**

Let $f_n:\{0,1\}^n\to\{0,1\}$ be one of the following Boolean functions:

(1) (Dictatorship: the first bit determines the outcome) $f_n^D(x_1,\dots,x_n)=x_1$;

(2) (Tribes) Partition $\{1,\dots,n\}$ into subsequent blocks of length $\log_2(n)-\log_2(\log_2(n))$ with perhaps some leftovers. Define $f_n^T(x_1,\dots,x_n)=\mathbf{1}_A$, where A is the event that at least one of the blocks consists of only 1's;

(3) (Iterated 3-majority function) Let $k\in\mathbb{N}$, consider a rooted 3-ary tree (the root vertex has degree 3, leaves degree 1 and other vertices degree 4) of depth k (in particular, there are $n=3^k$ leaves). To each leaf we assign 0 or 1, and apply the 3-majority function, i.e., $M(x_1,x_2,x_3)=\mathbf{1}_{\{\sum_i x_i>3/2\}}$, to determine the values of the vertices at depth $k-1$. We iterate this procedure until reaching the root, and define $f_n(x_1,\dots,x_n)$ to be the value at the root. Example: if $k=2$, we start with $x=(0,1,1;1,0,0;0,1,0)$, at depth 1 we get $(1,0,0)$, and hence, $f_n(x)=0$.

For all these examples, verify that f_n is monotone, compute the ith influence (for any i) and check "by hand" whether or not there is a phase transition.

Solution. The monotonicity follows directly from the definitions.

(1) $I_1(p)=\mathbb{P}_p[X_1\neq 1-X_1]=1$ and $I_i(p)=0$ for $i\geq 2$. Furthermore, $\mathbb{E}_p[f_n^D]=\mathbb{E}_p[X_1]=p$, which is linear, hence, there is no sharp phase transition;

(2) For simplicity write $g(n) := \log_2 n - \log_2 \log_2 n$. For the indices of leftovers (since A does not depend on them), $I_i(p) = 0$. These are $i > i_0 := \lfloor \frac{n}{g(n)} \rfloor g(n)$. For $i \leq i_0$, note that $\bar{X} \in A$, but $\bar{X}^{(i)} \notin A$ (X with ith coordinate flipped) is the event that the only block consisting of only 1's is the block containing index i. This event has probability $p^{\lfloor g(n) \rfloor}(1 - p^{\lfloor g(n) \rfloor})^{\lfloor \frac{n}{g(n)} \rfloor - 1}$. On the other hand, $\bar{X} \notin A$, but $\bar{X}^{(i)} \in A$ is the event that none of the blocks consist of only 1's, but the block containing ith element has all 1's except for the ith element which is 0. Its probability is $p^{\lfloor g(n) \rfloor - 1}(1-p)(1 - p^{\lfloor g(n) \rfloor})^{\lfloor \frac{n}{g(n)} \rfloor - 1}$. Therefore,

$$I_i(p) = \mathbb{P}_p[\bar{X} \in A, \bar{X}^{(i)} \notin A] + \mathbb{P}_p[\bar{X} \notin A, \bar{X}^{(i)} \in A] = p^{\lfloor g(n) \rfloor - 1}(1 - p^{\lfloor g(n) \rfloor})^{\lfloor \frac{n}{g(n)} \rfloor - 1}.$$

To determine whether there is a sharp phase transition, consider and compute

$$\mathbb{E}_p[f_n^T] = \mathbb{P}_p[A] = 1 - (\mathbb{P}_p[\text{all blocks have zeroes}])^{\lfloor \frac{n}{g(n)} \rfloor} = 1 - (1 - p^{\lfloor g(n) \rfloor})^{\lfloor \frac{n}{g(n)} \rfloor}.$$

We find that $p_c(n) = (1 - 2^{-1/\lfloor n/g(n) \rfloor})^{1/\lfloor g(n) \rfloor}$ (such that $\mathbb{E}_{p_c(n)}[f_n^T] = 1/2$). One can check that $p_c(n)$ converges to $1/2$ as n tends to infinity. Let ε_n be a positive sequence converging to zero (the rate of convergence is to be determined). Using Taylor's expansion of $\log(1 \pm x)$ at $x = 0$, we obtain

$$\begin{aligned}
\mathbb{E}_{p_c(n)(1+\varepsilon_n)}[f_n^T] &= 1 - \exp\Big(\Big\lfloor \frac{n}{g(n)} \Big\rfloor \log(1 - (1 - 2^{-1/\lfloor n/g(n) \rfloor})(1+\varepsilon_n)^{\lfloor g(n) \rfloor})\Big) \\
&\geq 1 - \exp\Big(- \Big\lfloor \frac{n}{g(n)} \Big\rfloor (1 - 2^{-g(n)/n})(1+\varepsilon_n)^{\lfloor g(n) \rfloor}\Big) \\
&\geq 1 - \exp\Big(- \frac{1}{2} \frac{n}{g(n)} (1 - (\log_2 n/n)^{1/n})(1+\varepsilon_n)^{g(n)}\Big).
\end{aligned}$$

Let us show that $\frac{n}{g(n)}(1 - (\log_2 n/n)^{1/n}) \to \log 2$ as n tends to infinity:

$$\begin{aligned}
\lim_n \frac{(1 - e^{\frac{1}{n}\log(\log_2 n/n)})}{g(n)/n} &= \log 2 \lim_n \frac{(1 - e^{\frac{1}{n}\log(\log_2 n/n)})}{\log n/n} \\
&\overset{\text{L'Hosp.}}{=} -\log 2 \lim_n e^{\frac{1}{n}\log(\log_2 n/n)} \lim_n \Big(\frac{1}{\log n} + \frac{\log\log n - \log n - \log\log 2}{\log n - 1}\Big) \\
&= \log 2 e^{\lim_n \frac{1}{n}\log(\log_2 n/n)} = \log 2.
\end{aligned}$$

Therefore, $\liminf_n \mathbb{E}_{p_c(n)(1+\varepsilon_n)}[f_n^T] \geq 1 - \exp(-\frac{\log 2}{2}\lim_n(1+\varepsilon_n)^{g(n)})$ with $\lim_n(1+\varepsilon_n)^{g(n)} = +\infty$ as long as $\varepsilon_n g(n) \to \infty$. For instance, take $\varepsilon_n = 1/\sqrt{g(n)}$, then $\liminf_n \mathbb{E}_{p_c(n)(1+\varepsilon_n)}[f_n^T] = 1$. Fully analogously, with the same choice of ε_n, one gets $\limsup_n \mathbb{E}_{p_c(n)(1-\varepsilon_n)}[f_n^T] = 0$. Hence, the model exhibits a sharp phase transition.

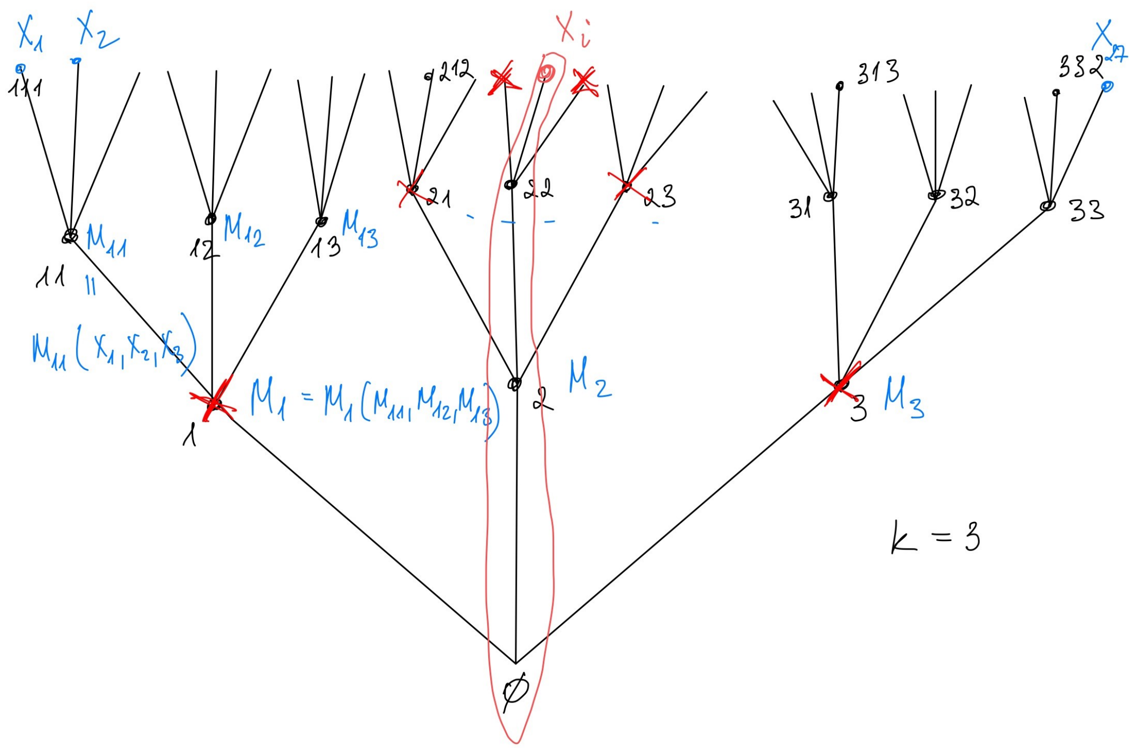
(3) Let us introduce some notations. We label vertices of the tree using Ulam-Harris labeling, meaning that the root has index $\emptyset$, the first generation has labels in $\mathbb{N}$ (in our case, just $1, 2$ and 3), in the next generation the labels are of the form $i1, i2, i3$ for ancestors of the vertex i, and so on (see the figure below). We further write M_u for the value at vertex u (u is an index as described above); note that for u, an index of length k (corresponding to leaves of the tree), $M_u = X_{1+\sum_{j=1}^k (u_j - 1)3^{k-j}}$, and that M_u is the majority function evaluated at M_{u1}, M_{u2}, M_{u3}. Note that $f_n(\bar{X}) \neq f_n(\bar{X}^{(i)})$ if and only if the swap of the value of X_i affects the change of the value at the root, which in turn can only happen if all vertices along the branch from the leaf to

the root have "siblings" taking two different values (on the figure, vertices marked with red crosses at each given generation must be of different values). Wlog (up to reindexing of the events), assume that the ith coordinate is exactly the leaf of the central branch, i.e., $i = \lfloor 3^k/2 \rfloor + 1$. Then, the above observation results into

$$I_i(p) = \mathbb{P}_p[M_1 \neq M_3, M_{21} \neq M_{23}, \dots, M_{22\dots21}(= X_{i-1}) \neq M_{22\dots23}(= X_{i+1})] = \prod_{j=1}^{k} P_j,$$

where P_j is the probability of $M_1 \neq M_3$ in a 3ary tree of depth j (note that $P_1 = \mathbb{P}_p[X_{i-1} \neq X_{i+1}]$). Here we used independence of sub-trees that arise when we erase the branch from the ith leaf to the root. Let us further write p_j for the probability of the event $M_1 = 1$ in a 3ary tree of depth j, then $P_j = 2p_j(1-p_j)$. It only remains to find p_j. To this end, notice that $M_1 = 1$ iff at least two of M_{11}, M_{12}, M_{13} are 1's; and since each M_{1l} has the law of M_1 in the tree of depth $j-1$, we obtain that $p_j = p_{j-1}^3 + 3p_{j-1}^2(1-p_{j-1})$. Altogether,

$$I_i(p) = \prod_{j=1}^{k}(2p_j(1-p_j)) \quad \text{with } p_j = p_{j-1}^3 + 3p_{j-1}^2(1-p_{j-1}) \text{ for } 2 \leq j \leq k, \quad p_1 = p.$$



Let us now move to the question of existence of a sharp phase transition. Observe first that for $p = 1/2$, $p_j = 1/2$ for all $j \leq k$, and $\mathbb{E}_p[f_n] = p_k^3 + 3p_k^2(1-p_k)(=: p_{k+1}) = 1/2$. Furthermore, note that $l \mapsto p_l$ is strictly increasing if $p_1 = p \in (1/2, 1)$, and strictly decreasing if $p \in (0, 1/2)$. Let us focus on the former case (the latter follows analogously): namely, assume that $p_1 = 1/2 + \varepsilon_n$ for some $\varepsilon_n > 0$. Let $\varepsilon > 0$, either for all n sufficiently large there exists $j_0(n, \varepsilon_n) < k = \log_3 n$ such that $p_{j_0} \geq 1 - \varepsilon$ (and by monotonicity, this holds also for all indices larger than j_0) — in this case we get as desired that $\mathbb{E}_{1/2+\varepsilon_n}[f_] = p_{k+1} \geq 1 - \varepsilon$ for all n sufficiently large; or there exists a subsequence $(n_l)_l$ (with abuse of notation still write just n) converging to infinity such that $p_k \leq 1 - \varepsilon$ (and hence all p_l for $l \leq k$). In the latter case, on the one hand, $p_{k+1} \leq 1 - 3\varepsilon^2 + 2\varepsilon^3$. On the other hand, $p_{k+1} = p \prod_{j=1}^{k} p_{j+1}/p_j = p \prod_{j=1}^{k} p_j(3 - 2p_j) \geq (1/2 + \varepsilon_n)(1 + \varepsilon_n - 2\varepsilon_n^2)^k$, where we used that $h : x \in [0, 1] \mapsto x(3 - 2x)$ is increasing on $[0, 3/4]$ and decreasing on $[3/4, 1]$ (and so, $\min_{x \in [1/2+\varepsilon_n, 1-\varepsilon_n]} h(x) = \min(h(1/2+\varepsilon_n), h(1-\varepsilon_n)) = (1-\varepsilon_n)(1+2\varepsilon_n))$. By choosing $\varepsilon_n = 1/\sqrt{\log n}$, for instance, we get that $1/2(1+(1-o(1))/\sqrt{\log n})^{\log_3 n} \gg 1$, which is clearly a contradiction since p_{k+1} must be smaller than 1. Therefore, the model exhibits a sharp phase transition with $p_c = 1/2$ and $\varepsilon_n \geq 1/\sqrt{\log n}$ (not optimal).

$\square$