Exercise 4 ("Open question"). *Find classes of functions $f_n : \mathbb{R}^n \to \mathbb{R}$ such that $f_n \in C^\infty$ and so that*

$$\sup_x \left| \frac{\partial^3}{\partial x_i^3} f_n(x) \right| = o(n^{-1}) \quad \forall 1 \le i \le n,$$

for all n sufficiently large.

Solution: (1) We start by looking at polynomials in n variables $p_n : \mathbb{R}^n \to \mathbb{R}$,

$$p_n(\underline{x}) = \sum_{i_1=0}^{M_1} \cdots \sum_{i_n=0}^{M_n} a_{i_1 \dots i_n} \, x_1^{i_1} \cdots x_n^{i_n}$$

Viewed as a fct in variable x_i, it can be rewritten as $\sum_{j=0}^{M_i} Q_j\left((x_k)_{k \neq i}\right) x_i^j$ for appropriate polynomials in $n-1$ variables Q_j

$$\partial_i \, p_n(\underline{x}) = \sum_{j=1}^{M_i} Q_j \, j \, x_i^{j-1}$$

$$\partial_{ii} \, p_n(\underline{x}) = \sum_{j=2}^{M_i} Q_j \, j(j-1) \, x_i^{j-2}$$

$$\partial_{iii} \, p_n(\underline{x}) = \sum_{j=3}^{M_i} Q_j \, j(j-1)(j-2) \, x_i^{j-3}$$

So, for p_n to satisfy $\sup_x |\partial_{iii} p_n| = o(n^{-1})$, it may not have non-trivial summands with x_i to power higher than 3 ($\rightsquigarrow M_i \overset{!}{\le} 3$) (other $\sup_{x_i} |\partial_{iii} p_n| = \infty$)

Furthermore, if $M_i = 3$

$$|\partial_{iii} p_n(\underline{x})| = 3! \, |Q_3|$$

In particular, Q_3 may not have non-zero powers of x_j's $(j \neq i)$ since other $\sup_{(x_j)_{j\neq i}} |Q_3| = \infty$

$$\Rightarrow \quad p_n(\underline{x}) \overset{!}{=} o(n^{-1}) \, x_i^3 + \underbrace{\sum_{j=0}^{2} Q_j\left((x_k)_{k \neq i}\right) x_i^j}_{\partial_{iii}(\;) = 0}$$

But this should hold for any i, hence,

$$\sup_x |\partial_{iii} p_n(x)| = o(n^{-1}) \quad \forall i \quad \text{iff,}$$

$$p_n(\underline{x}) = \sum_{i=1}^{n} o(n^{-1})\, x_i^3 + \sum_{i_1=0}^{2} \dots \sum_{i_n=0}^{2} a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n}$$

(2) Consider functions of the form $g \circ p_n : \mathbb{R}^n \to \mathbb{R}$ for some $g \in C^3(\mathbb{R})$ and some poly in n-variables

Then, $\partial_{iii}\, g \circ p_n = g'''(p_n)(\partial_i p_n)^3 + 3g''(p_n)(\partial_i p_n)(\partial_{ii} p_n)$
$+ g'(p_n)\, \partial_{iii} p_n \qquad \star$

Note that we can assume wlog that $g''' \not\equiv 0$ (since other $g(y) = cy^2 + by + d$ and we're back to the polynomial setting discussed in (1))

By our assumption on g, p_n, the unbddness of $\star$ can only appear at $|x_i| \to \infty$ for some i. ($\Rightarrow |p_n(x)| \to \infty$ unless $M_i = 0$ for general x)

(A) Consider first the case $\max_i M_i \leq 1$:

- for $\max_i M_i = 0$, we just have $g \circ p_n \equiv cst \to$ incl. in (2)
- $\max_i M_i = 1 \Rightarrow \partial_{iii} p_n = \partial_{ii} p_n = 0 \quad \forall i$

Only need to control $|g'''(p_n)\, (\partial_i p_n)^3|$ $\forall i$ with $M_i = 1$ (when $|p_n| \to \infty$)

$$= \sum_{\substack{k_j = 0 \\ \text{for } j \neq i}}^{1} a_{k_1 \dots 1 \dots k_n} \left(\prod_{j \neq i} x_j^{k_j} \right) =: Q_1^i(x^{\setminus i})$$

at ith position; $(x_j)_{j \neq i}$

$Q_1^i(x^{\setminus i})\, x_i + Q_0^i(x^{\setminus i})$

there are two different cases:

(I) $Q_1^i(\hat{x}^i) = O(1)$ unif. $\forall i$

$\Longleftrightarrow \quad p_n = \sum_k a_k x_k + cst$

$|g'''(p_n)\, a_i^3| \overset{!}{=} o(n^{-1})$

$\rightsquigarrow$ Let $g_0 \in C^3$ with $\|g'''\|_\infty \le C$, then the following option would give the desired result

- $\underbrace{b_n g_0}_{\overset{!!}{g}}\left(\sum_{k=1}^n a_k x_k + cst\right)$ s.t. $b_n \in \mathbb{R}$, $\max_{k \ge 1}^n |a_k| = o\left(\frac{1}{|n b_n|^{1/3}}\right)$

(For instance, in class you had an example with $b_n = 1$, $a_k = \frac{1}{\sqrt{n}}$ $\forall k$)

(II) There exists some i s.t.

$\sup_x |Q_1^i(\hat{x}^i)| = \infty$

Note that in this case

$|g'''(\overbrace{Q_1^i(\hat{x}^i)\, x_i + Q_0^i(\hat{x}^i)}^{p_n(x)})\, Q_1^i(\hat{x}^i)|$ can be unif bdd, only if when $|Q_1^i(\hat{x}^i)| \xrightarrow[\text{along subseq.}]{|\hat{x}^i| \to \infty} \infty$ also $\inf_{x_i} |p_n(x)|$ tends to infinity and $g'''(y) \xrightarrow{|y| \to \infty} 0$ sufficiently quickly. Note, however, that by choosing $x_i = -\frac{Q_0^i(\hat{x}^i)}{Q_1^i(\hat{x}^i)}$, we get

$|\underbrace{g'''(0)}_{cst}\, Q_1^i(\hat{x}^i)| \longrightarrow \infty$ along some subseq $|\hat{x}^i| \to \infty$

$\Longrightarrow$ If $\max_i M_i = 1$, the only possibility is

(B) Suppose that $\max_i M_i \geq 2$, in this case we have to control at least two terms for each i with $M_i \geq 2$

$$|g'''(p_n)(\partial_i p_n)^3| \quad \text{and} \quad |g''(p_n)(\partial_i p_n)(\partial_{ii} p_n)|$$

and potentially $|g'(p_n)\partial_{iii} p_n|$

Analogously to the above,

$$|g'''(p_n)(\partial_i p_n)^3| = \left| g'''\left(\sum_{j=0}^{M_i} Q_j^i(\hat{x}^i) x_i^j\right) \left(\sum_{j=1}^{M_i} Q_j^i(\hat{x}^i) j x_i^{j-1}\right)\right|$$

The question is similar to before, whether for some i there exists a subseq x_k s.t. $|x_k| \xrightarrow{k\to\infty} \infty$ and $\left|\sum_{j=1}^{M_i} Q_j^i((\hat{x}_k^i)) j (x_k)_i^{j-1}\right| \longrightarrow \infty$, but $\sum_{j=0}^{M_i} Q_j^i(\hat{x}_k^i)(x_k)_i^j$ stays bdd.

> More generally, if there exists a subseq of $|x| \to \infty$ s.t. $\max_i \left(|(\partial_{ii} p_n)(\partial_i p_n)| \vee |\partial_{iii} p_n| \vee |\partial_i p_n|^3\right) \longrightarrow \infty$, but p_n stays bdd along this subseq., then there's no $g \in C^3$ s.t. $g \circ p_n$ satisfies the desired condition on its 3rd order derivative

I don't know how to conclude in the general case, but let's consider a few examples of $\max_i M_i \geq 2$ that work

- $p_n(x) = \sum_{k=0}^{M_1} a_k x_1^k \quad (\text{or any other fixed index})$, $a_{M_1} \neq 0$

  $\rightsquigarrow$ In this case, unbddness can only come from $|x_1| \to \infty$, and so, it suffices to understand the behav. of the leading term of p_n

$$\left|g'''(p_n)\,(\partial_i p_n)^3\right| \underset{|x_1|\to\infty}{\sim} \left|g'''\left(a_{M_1} x_1^{M_1 (1+o(1))}\right)\underbrace{\left(a_{M_1} M_1 x_1^{M_1-1 (1+o(1))}\right)^3}_{M_1\, a_{M_1}^{3/M_1} \left(a_{M_1} x_1^{M_1}\right)^{3-\frac{3}{M_1}}}\right|$$

$$\left|g''(p_n)\,(\partial_i p_n)(\partial_{ii} p_n)\right| \sim \left|g''\left(a_{M_1} x_1^{M_1 (1+o(1))}\right)\underbrace{a_{M_1}^2 M_1^2 (M_1-1)\, x_1^{2M_1-3 (1+o(1))}}_{M_1^2(M_1-1)\, a_{M_1}^{3/M_1}\left(a_{M_1} x_1^{M_1}\right)^{2-\frac{3}{M_1}}}\right|$$

$$\left|g'(p_n)\,(\partial_{iii} p_n)\right| \sim \left|g'\left(a_{M_1} x_1^{M_1 (1+o(1))}\right)\underbrace{a_{M_1} M_1 (M_1-1)(M_1-2)\, x_1^{M_1-3 (1+o(1))}}_{a_{M_1}^{3/M_1} M_1 (M_1-1)(M_1-2) \left(a_{M_1} x_1^{M_1}\right)^{1-\frac{3}{M_1}}}\right|$$

$\rightarrow$ choose $g \in C^3$ s.t. $g(y) \overset{|y|\to\infty}{=} O\left(|y|^{3/M_1}\right)$ with nice derivatives

$$g^{(k)}(y) \underset{|y|\to\infty}{\sim} O\left(|y|^{\frac{3}{M_1}-k}\right) \rightarrow \text{no blow up in } f \text{ at } \infty$$

For instance, $g(y) \overset{|y|\to\infty}{=} |ay+b|^\alpha$ for $\alpha \le 3/M_1$, $a,b \in \mathbb{R}$ (also negative powers)

or $g(y) \overset{|y|\to\infty}{=} \dfrac{|ay+b|^\alpha}{\log^\alpha |cy+d|}$ $\quad \underbrace{\log\log\log\ldots\log}_{k}|y|$

or $g \in C_c^3$ work

$\longrightarrow$ Our options to obtain the desired result in this case ($M_i \ge 2$ for one fixed i)

Let $g \in C^3$ such that (in particular, $\|g''\|_\infty, \|g'''\|_\infty < \infty$, ($\|g'\|_\infty < \infty$ if $M_i \ge 3$))

as $|y| \to \infty$, $g^{(k)}(y) = O\left(|y|^{3/M_i - k}\right)$ $\quad k = 0,1,2,3$ (see examples above)

$$f_n(x) := o(n^{-1}) \cdot g\left(\sum_{\ell=0}^{M_i} a_\ell x^\ell\right), \quad a_\ell \in \mathbb{R} \text{ fixed}, \; a_{M_i} \neq 0$$

or alternatively,

$$f_n(\underline{x}) := g\left(\sum_{\ell=0}^{M_i} a_\ell(n)\, x_i^\ell\right) \text{ with}$$

with a_ℓ's chosen depending on n

$\longrightarrow$ for each g, a specific choice has to be made

For instance, if $a_\ell = 0$ for $\ell \neq M_i$

and $g = \begin{cases} 1 \text{ on } [-1+\varepsilon, 1-\varepsilon] \\ |y|^\alpha \text{ on } [-1-\varepsilon, 1+\varepsilon]^c \text{ for } \alpha \leq 3/M_1 \\ \text{smooth interpolation with bdd deriv up to 3rd order on} \end{cases}$

$\rightsquigarrow f_n(\underline{x}) = g\left(a_{M_i} x_i^{M_i}\right)$ with $a_{M_i}^{3/M_i} = o(n^{-1})$

One could adjust the above reasoning for it to apply to the case $p_n = \sum_{k=1}^{n} \sum_{\ell=0}^{2M_k-1} a_{k\ell}\, x_k^\ell \pm \sum_{k=1}^{n} |a_{k(2M_k)}|\, x_k^{2M_k}$ (for $\max_i M_i \geq 1$)

(note that the choice of highest order term with even powers for all x_k and with the same sign of coeffs prevents existence of a sequence in $\boxed{}$ if $x_i \to \infty$, $|p_n(x)| \not\to \infty$ indep of $(x_j)_{j\neq i}$)

$\longrightarrow$ Take $f_n = g\left(p_n(\underline{x})\right)$, $o(n^{-1})$

with $g \in C^3$ s.t. $g^{(k)}(y) \overset{|y|\to\infty}{=} O\left(|y|^{\frac{3}{2\max_\ell M_\ell} - k}\right)$ for $k = 0, 1, 2, 3$

$\sup_n \max_{k,\ell} |a_{k\ell}| < \infty$, $a_{k(2M_k)} \neq 0 \ \forall k$