TOPICS IN PROBABILITY. PART II: UNIVERSALITY

Exercise sheet 9: Universality and semicircle law

Exercise 1 (Moments of semicircle law)**.** *;*
Let μ be the semicircle law, i.e., $\mu(\mathrm{d}x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{\{|x|\le 2\}}$. Show that for $k \in \mathbb{N}_0$,

$$\mu[x^k] = \begin{cases} 0 & k \text{ odd;} \\ C_{k/2} & k \text{ even,} \end{cases}$$

where $C_k = \frac{1}{1+k}\binom{2k}{k}$ is the kth Catalan number.

Exercise 2 (Trace, operator and Frobenius norms)**.** *;*
Let $(A^i)_{i=1}^k$ be matrices of sizes $m_{i-1} \times m_i$ such that $m_0 = m_k$. Show that for any $1 \le i < j \le k$,

$$|\mathrm{Trace}(A^1 A^2 \dots A^k)| \le \|A^i\|_F \|A^j\|_F \prod_{l: l\neq i,j} \|A^l\|_{op},$$

where $\|A\|_F = \sqrt{\mathrm{Trace}(\bar{A}^t A)} = \sqrt{\sum_{i,j} |a_{ij}|^2}$ is the Frobenius norm and $\|A\|_{op} = \sup_{\|x\|_2=1} \|Ax\|_2$ the operator norm.

Exercise 3 (Bound on difference of eigenvalues)**.** *;*
Let A, B be two Hermitian $n \times n$ matrices. Let us denote their eigenvalues $(\lambda_i^A)_{i=1}^n$ and $(\lambda_i^B)_i$, respectively. Show that

$$\inf_\sigma \sum_{i=1}^n |\lambda_i^A - \lambda_{\sigma(i)}^B|^2 \le \|A-B\|_F^2,$$

where the infimum is taken over all permutations of n elements.

Exercise 4 (Bound on the operator norm of Wigner matrix with bounded entries)**.** *;*
Let X be a Wigner $N \times N$ matrix with mean-zero, uniformly bounded (wlog, by one) entries, i.e., $(X_{ij})_{i \le j \le N}$ are independent with $\mathbb{E}[X_{ij}] = 0$ and $|X_{ij}| \le 1$ almost surely and $X_{ji} = \bar{X}_{ij}$. Show that there exists $C, c > 0$ (absolute constants independent of N) such that $\mathbb{P}[\|X\|_{op}/\sqrt{N} \ge t] \le e^{-cNt^2}$ for all $t \ge C$.

You may proceed as follows:

1. *Show (using an appropriate concentration of measure inequality you know) that for any unit vector $x \in \mathbb{C}^N$ and $N \times N$-matrix $\tilde{X}$ with (all!) mutually independent, uniformly bounded by one entries of mean zero, $\mathbb{P}[|\tilde{X}x|/\sqrt{N} > t] \le e^{-cNt^2}$ for all $t \ge C$ for some appropriate absolute constant $C > 0$;*
2. *Prove that $\mathbb{P}[\|\tilde{X}\|_{op}/\sqrt{N} > t] \le \mathbb{P}[\bigcup_{x\in G}\{|\tilde{X}x|/\sqrt{N} > t/2\}]$ for a maximal $1/2$-net $G \subset \mathbb{C}^N \cap \mathbb{S}^{2N-1}$, i.e., all points of G are separated by a distance at least $1/2$, and G is maximal w.r.t. set-inclusion;*
3. *Estimate the number of points of G appropriately and conclude the result for $\|\tilde{X}\|_{op}/\sqrt{N}$;*

(4) Conclude for $\|X\|_{op}/\sqrt{N}$.