TOPICS IN PROBABILITY. PART II: UNIVERSALITY

Exercise sheet 7: Universality and CLT

Exercise 1. *Does the CLT remain true if we do not assume that variances of the variables are finite? More precisely, let $(X_i)_i$ be i.i.d. random variables with mean zero and infinite variance. Does $\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i$ converge in law (to normal distribution)?*

$\star$: *Suppose X_i's are symmetric with $\mathbb{P}[|X_i| > x] = x^{-\alpha}$ for $\alpha \in (0, 2)$ for all $x \geq 1$. For which α's is X_i integrable? Is it possible to find a different normalization of $\sum_{i=1}^n X_i$ such that the resulting variable converges in law (to some probability law)? What is the intuitive explanation?*

Exercise 2 (Lindeberg-Feller CLT)**.** *Let $(X_i)_i$ be independent random variables with $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = \sigma_i^2 < \infty$ (not necessarily equal to one another). Let $s_n^2 \coloneqq \sum_{i=1}^n \sigma_n^2$. Show by adjusting the proof of Lindeberg exchange principle to this more general setting that if for any $\varepsilon > 0$,*

$$\frac{1}{s_n^2}\sum_{i=1}^n \mathbb{E}[X_i^2 \mathbf{1}_{\{|X_i|>\varepsilon s_n\}}] \xrightarrow{n\to\infty} 0,$$

then $\frac{1}{s_n}\sum_{i=1}^n X_i$ converges in law to a standard normal variable.
Hint: Take Y_i's (in the proof of the exchange principle) to be independent centered normal with variance σ_i^2, note that now $\tilde{X}_k$'s are not identically distributed — how does it affect (1.2)?

Find a sequence of independent random variables $(X_i)_i$ with mean zero and variance one such that $\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i$ does not converge in law to a standard normal variable.

Exercise 3 (Sherrington-Kirkpatrick ground state)**.**

Let us consider statistical mechanics model of n spins, i.e. particles that can be in one of two states $\{\pm 1\}$, due to Sherrington and Kirkpatrick, which models a rough energy landscape by introducing random interactions between the spins. More precisely, for a configuration of spins $\sigma \in \{-1, 1\}^n$, let the energy $H(\sigma)$ be defined as

$$H(\sigma) = \frac{1}{n^{3/2}} \sum_{1\leq i<j\leq n} X_{ij}\sigma_i\sigma_j,$$

where X_{ij} are independent with zero mean and unit variance. The ground-state energy, that is, the energy the system attains at zero temperature (when in thermal equilibrium), is given by

$$Z = \min_{\sigma\in\{-1,1\}^n} H(\sigma).$$

One of the basic questions to ask is whether Z is universal, more precisely, whether it depend significantly on the distribution of X_{ij}'s or not? Universality is important from the physical perspective: it states that macroscopic observations are insensitive to the microscopic details in the description of physical systems. We want to apply Universality theorem proven in the *lecture, but Z is not three times differentiable w.r.t. X_{ij}. The solution is to introduce a suitable smooth approximation of the minimum.*

- *Show that for any $\beta > 0$,*

$$|Z - Z_\beta| \le \frac{n \log 2}{\beta}, \quad Z_\beta = -\frac{1}{\beta} \log \left(\sum_{\sigma \in \{-1,1\}^n} e^{-\beta H(\sigma)} \right).$$

- *Combine part one with the universality theorem to show that the expected ground-state energy $\mathbb{E}[Z]$ is insensitive to the distribution of the variables X_{ij}. Assume that the third moments of X_{ij}'s exist and are uniformly bounded.*
- $\star$*: extend to the case of unbounded/non-existent third moments.*