TOPICS IN PROBABILITY. BETWEEN PARTS I AND II: RANDOM PROJECTIONS, FIRST STEPS TOWARDS UNIVERSALITY

Exercise sheet 6: log-Sobolev inequality and random projections

Exercise 1 (Proof of Gaussian log-Sobolev inequality). *(1) Complete the proof (sketched in class) of Gaussian log-Sobolev inequality*

$$\mathrm{Ent}[f] \leq \frac{1}{2}\mathbb{E}[\|\nabla f\|^2/f]$$

under the assumption on boundedness of derivatives of $f > 0$ up to third order by filling in the missing details.

(2) Extend the result to continuously differentiable functions $f \geq 0$.

Exercise 2 (log-Sobolev inequality for general Gaussians). *Let $X \in \mathbb{R}^n$ has centered Gaussian distribution with covariance matrix Ξ. Show that for any continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ the following holds,*

$$\mathrm{Ent}[f^2] \leq 2\mathbb{E}[\langle \Xi\nabla f(X), \nabla f(X)\rangle].$$

Exercise 3 (log-Sobolev implies Poincaré). *Prove that Gaussian log-Sobolev inequality (for standard Gaussian vector) implies Gaussian Poincaré inequality.*

Hint: Let $\varepsilon > 0$ be small and use the log-Sobolev inequality for $(1 + \varepsilon f)$. Show that $\mathrm{Ent}[(1 + \varepsilon f)^2] = 2\varepsilon^2\mathrm{Var}[f(X)] + \mathcal{O}(\varepsilon^3)$.

Exercise 4 (Weak Poincaré lemma). *Let $m \in \mathbb{N}$ be fixed. Consider a random vector X^N uniformly distributed on the unit sphere $S^{N-1} \subset \mathbb{R}^N$. Let $X^{m,N}$ denote the vector consisting of its first m coordinates. Prove that as N tends to infinity the law of $\sqrt{N}X^{m,N}$ converges to standard Gaussian distribution in dimension m.*

Exercise 5 (Almost isometric projection of uniformly distributed point on the sphere). *Let N be very large and let $S^{N-1} \subset \mathbb{R}^N$ be a unit $N-1$-sphere. Let X be a point chosen uniformly on S^{N-1} and $T : \mathbb{R}^N \to \mathbb{R}^m$ be a projection on the first m coordinates. Find a suitable normalization of T by some power of $\frac{m}{N}$ so that the following holds $1 - \varepsilon \leq \|c_{norm}TX\|_2 \leq 1 + \varepsilon$[1] with probability at least $1 - 4e^{-c\varepsilon^2 m}$ for some uniform constant $c > 0$. Here c_{norm} is some power of $\frac{m}{N}$, which you need to find.*

You might proceed as follows

- *Find the normalization constant by computing L^2-norm of Tx and observing that for the normalized operator it has to be equal to 1 (why?). Check yourself* [2].
- *Let Y be a standard N-dimensional Gaussian vector. Prove that X has the same law as $Y/\|Y\|_2$. Conclude that $\|Y\|_2 X$ has the law of N-dimensional standard Gaussian vector.*

[1] note that $\|X\|_2 = 1$

[2] $c_{\mathrm{norm}} = \sqrt{\frac{N}{m}}$

- *To prove that you actually get the "almost isometry" property for the normalized operator* $c_{norm}T$ *compare* $c_{norm}TX$ *and* $\frac{1}{\sqrt{m}}T(\|Y\|_2 X)$ *(what law does the latter variable has?).*
- *Additionally to the last step: estimate the concentration probability of the norm of m-dimensional standard Gaussian around* $\sqrt{m}$*. Chernoff-type bounds might be helpful.*

Exercise 6 (log-Sobolev for Rademacher random variables)**.** *Let* $X_1, \ldots, X_n$ *be i.i.d. symmetric Rademacher random variables,* $f : \mathbb{R}^n \to \mathbb{R}$*. Show that*

$$\mathrm{Ent}[f^2] \leq \sum_{i=1}^{n} \mathbb{E}[(f - f^{(i)})^2],$$

where $f = f(X_1, \ldots, X_n)$ *and* $f^{(i)} = f(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n)$ *with* X_i' *being an independent copy of* X_i*.*

You may proceed as follows:

(1) Use tensorization of entropy to reduce to a one-dimensional problem;

(2) Verify the following inequality and prove that it yields the desired result,

$$\forall a, b \in \mathbb{R} : \quad \frac{a^2}{2} \log a^2 + \frac{b^2}{2} \log b^2 - \frac{a^2 + b^2}{2} \log \frac{a^2 + b^2}{2} \leq \frac{(a - b)^2}{2}.$$