TOPICS IN PROBABILITY. PART I: CONCENTRATION

Exercise sheet 3: Efron-Stein and Poincaré inequalities, Subgaussianity

1. Efron-Stein and Poincaré inequalities

Exercise 1 (Extending Gaussian Poincaré to C^1 functions)**.**

Let $f : \mathbb{R} \to \mathbb{R}$ be any continuously differentiable function and X be a standard normal random variable. Show that

$$\mathrm{Var}[f(X)] \leq \mathbb{E}[(f'(X))^2].$$

Exercise 2 (Gaussian Poincaré for Lipschitz functions is better than direct Efron-Stein)**.**

Let X be a standard Gaussian vector, f be L-Lipschitz function and set $Z = f(X)$. Compare the bounds for $\mathrm{Var}[Z]$ *obtained using Gaussian Poincaré inequality to the ones that you get by applying Efron-Stein inequality (in the most obvious way). What important feature does Poincaré inequality have compared to Efron-Stein in this situation?*

(1) Prove that for any Gaussian vector Y,

$$\mathrm{Var}[\max_{i=1}^{n} Y_i] \leq \max_{i=1}^{n} \mathrm{Var}[Y_i]$$

(2) Recall Exercise 2 Sheet 2. Use Gaussian Poincaré inequality to prove that

$$\sqrt{D} - 1 \leq \mathbb{E}[Z],$$

where Z be a non-negative random variable such that Z^2 has a chi-squared distribution with D degrees of freedom.

Can you get these bounds using Efron-Stein inequality?

Exercise 3 (Poisson Poincaré inequality)**.**

Let $f : \mathbb{N}_0 \to \mathbb{R}$ be a real-valued function defined on non-negative integers, denote its discrete derivative as $Df(x) = f(x+1) - f(x)$. Let X be a Poisson random variable with intensity μ. Prove that

$$\mathrm{Var}[f(X)] \leq \mu \mathbb{E}[(Df(X))^2].$$

Hint: infinite divisibility of Poisson distribution and Efron-Stein inequality might be useful.

2. Subgaussianity

Exercise 4 (Subgaussian properties)**.**

Let X be a random variable. Show that the following properties are equivalent so that the parameters $C_i > 0$ appearing in the properties below differ from each other by an absolute constant factor, meaning that there exists $C > 0$ such that property i implies property j with parameter $C_j \leq CC_i$ for all $i, j = 1, \ldots, 5$

(1) The tails of X satisfy

$$\mathbb{P}[|X| \geq t] \leq 2e^{-t^2/C_1^2} \quad \textit{for all } t \geq 0.$$

(2) The moments of X satisfy

$$\|X\|_{\mathrm{L}^p} \leq C_2\sqrt{p} \quad \text{for all } p \geq 1.$$

(3) The MGF of X^2 satisfies

$$\mathbb{E}\left[e^{\lambda^2 X^2}\right] \leq e^{C_3^2\lambda^2} \quad \text{for all } |\lambda| \leq \frac{1}{C_3}.$$

(4) The MGF of X^2 is bounded at some point, namely

$$\mathbb{E}\left[e^{X^2/C_4^2}\right] \leq 2.$$

Moreover if $\mathbb{E}[X] = 0$, then the above properties are also equivalent to

(5) The MGF of X satisfies

$$\mathbb{E}\left[e^{\lambda X}\right] \leq e^{C_5^2\lambda^2} \quad \text{for all } \lambda \in \mathbb{R}.$$

More precisely, show that

- $(1) \Rightarrow (2)$ *with* $C_2 \geq \sqrt{\pi}C_1$;
- $(2) \Rightarrow (3)$ *with* $C_3 \geq 2\sqrt{e}C_2$;
- $(3) \Rightarrow (4)$ *with* $C_4 \geq C_3/\sqrt{\log 2}$;
- $(4) \Rightarrow (1)$ *with* $C_1 \geq C_4$;
- $(3) \Rightarrow (5)$ *with* $C_5 \geq C_3$ *(under mean zero assumption);*
- $(5) \Rightarrow (1)$ *with* $C_1 \geq 2C_5$ *(under mean zero assumption).*

A random variable satisfying one of the above equivalent properties is called ***subgaussian****. To be more specific, we will call a random variable* σ^2***-subgaussian*** *if property* (5) *holds for* $X - \mathbb{E}[X]$ *with* $C_5^2 = \sigma^2/2$*. The constant* σ^2 *is called the* ***variance proxy****. The smallest such* σ^2 *is called the* ***optimal variance proxy****.*

Exercise 5 (Hoeffding lemma)**.**

Let $a \leq X \leq b$ a.s. for some $a, b \in \mathbb{R}$. Then $\mathbb{E}\left[e^{\lambda(X-\mathbb{E}[X])}\right] \leq e^{\lambda^2(b-a)^2/8}$. That is, X is $(b-a)^2/4$-subgaussian.

Hint: consider function $\psi(\lambda) = \log\mathbb{E}\left[e^{\lambda(X-\mathbb{E}[X])}\right]$. Show that $\psi''(\lambda)$ can be interpreted as a variance of $X - \mathbb{E}[X]$ with respect to a new appropriately chosen probability measure. Use some suitable bound for variances that you know (sheet 2 might be helpful).