TOPICS IN PROBABILITY. PART I: CONCENTRATION

Exercise sheet 1: Basic concentration inequalities

Exercise 1 (Recap: Markov's inequality and direct corollaries).

- *Suppose that X is a random variable, $a > 0$. Show that $\mathbb{P}[|X| \geq a] \leq \frac{\mathbb{E}[|X|]}{a}$.*
- *Prove extended version of Markov's inequality. Namely, if $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is monotonically increasing function, then $\mathbb{P}[|X| \geq c] \leq \frac{\mathbb{E}[\phi(|X|)]}{\phi(c)}$ for all $c > 0$ such that $\phi(c) \neq 0$.*
- *Conclude Chebyshev's inequality. More precisely, assume that X is an integrable random variable with mean μ and a finite non-zero variance σ^2, $a > 0$ and show that $\mathbb{P}[|X - \mu| \geq a] \leq \frac{\sigma^2}{a^2}$. Recall that the variance is given by $\sigma^2 = \mathbb{E}[(X - \mu)^2]$.*
- *(Exponential inequalities) Let X be a random variable, $b \in \mathbb{R}$. Prove (using Markov's inequality applied to suitable non-negative random variables) that $\mathbb{P}[X \geq b] \leq e^{-tb} M(t)$ for all $t \geq 0$ and $\mathbb{P}[X \leq b] \leq e^{-tb} M(t)$ for all $t \leq 0$, where $M(t) = \mathbb{E}[e^{tX}]$ is the moment generating function (MGF) of X.*
  Note that we can optimize the bounds by taking in the above inequalities infimum over $t \geq 0$, $t \leq 0$, respectively. The resulting inequalities are called Chernoff bounds in some literature (sometimes more specific results are called that, see Exercise 4).

Exercise 2 (p-moments via tails and applications).

Let X be a random variable, $p \in (0, \infty)$. Show that

$$\mathbb{E}[|X|^p] = \int_0^\infty p t^{p-1} \mathbb{P}[|X| > t] \mathrm{d}t$$

whenever the rhs is finite.

Suppose now that X is a random variable satisfying $\mathbb{P}[|X| \geq t] \leq 2e^{-bt}$ for some $b > 0$. Show that $\mathbb{E}[|X|^n] \leq 2b^{-n} n!$ and conclude that $\|X\|_{\mathrm{L}^n} \leq \frac{cn}{b}$ for some uniform constant $c > 0$ (Stirling's formula might be helpful).

Exercise 3 (Tails of normal distribution).

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ be a Gaussian random variable.

- *Using Chernoff bounds (optimized exponential inequalities) estimate $\mathbb{P}[X - \mu \geq c]$ for $c \geq 0$.*
- *Now suppose that that $\mu = 0, \sigma^2 = 1$. Show that for all $c > 0$,*

$$\left(\frac{1}{c} - \frac{1}{c^3}\right) \frac{1}{\sqrt{2\pi}} e^{-c^2/2} \leq \mathbb{P}[X \geq c] \leq \frac{1}{c} \frac{1}{\sqrt{2\pi}} e^{-c^2/2}.$$

  Hint: use density function, for the lower bound $g(x) = (1 - 3x^{-4}) \mathbf{1}_{x > c} \leq 1$ might be helpful.

Compare bounds of first and second bullet points in the case $\mu = 0, \sigma^2 = 1$.

Exercise 4 (Large deviations from LLN, Chernoff).

Let $X_i \in \mathrm{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ *be iid random variables with expectation* μ *and MGF* $M(t) = \mathbb{E}[e^{tX_1}]$. *Let* $S_n := \sum_{i=1}^n X_i$. *Show that for all* $n \in \mathbb{N}$ *and* $a \in \mathbb{R}$,

$$\begin{cases} \mathbb{P}\left[\frac{S_n}{n} \geq a\right] \leq e^{-nI(a)} & \text{if } a \geq \mu, \\ \mathbb{P}\left[\frac{S_n}{n} \leq a\right] \leq e^{-nI(a)} & \text{if } a \leq \mu, \end{cases}$$

where $I(a) := \sup_{t \in \mathbb{R}}(at - \log M(t))$. *Note that the supremum is taken over all* $\mathbb{R}$.
You can follow the following steps:

- *Reduce to the case* $\mu = 0$;
- *Exponential Markov inequality;*
- *Use Jensen's inequality to conclude that the supremum in* $I(a)$ *can be taken over all of* $\mathbb{R}$.