TOPICS IN PROBABILITY. PART III: PHASE TRANSITION

Exercise sheet 11: Sharp phase transition and Hypercontractivity

Exercise 1 (Russo's almost $0-1$ law)**.** *Consider a product measure $\mathbb{P}_p$ on $\{0,1\}^n$: each 1 is assigned weight p. Prove the following result: for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any increasing event A with the associated ith influence satisfying $I_i^p(A) (= \mathbb{P}_p[\mathbf{1}_A(\bar{X}) \neq \mathbf{1}_A(\bar{X}^{(i)}]) \leq \delta$ for all i, p, there exists p_c such that for all $p \geq p_c + \varepsilon$, $\mathbb{P}_p[A] \geq 1 - \varepsilon$, and for all $p \leq p_c - \varepsilon$, $\mathbb{P}_p[A] \leq \varepsilon$.*

Exercise 2 (Heat semigroup on the hypercube)**.**

Let $\mu = \mu_1 \otimes \ldots \otimes \mu_n := \left(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1\right)^{\otimes n}$ be the uniform measure on the hypercube $\{-1,1\}^n$. Define the process $X_t = (X_t^1, \ldots, X_t^n)_{t \geq 0}$ as follows. To each coordinate i, we attach an independent Poisson process $(N_t^i)_{t\geq 0}$ of intensity one[1]*. Then,*

- *draw $X_0 \sim \mu$ independently from the PP $N = (N^1, \ldots, N^n)$;*
- *each time N_t^i jumps for some i, replace the current value of X_t^i by an independent sample from μ_i while keeping the remaining coordinates fixed.*

We define $P_t f(x) := \mathbb{E}[f(X_t)|X_0 = x]$. Prove the following properties of $(X_t)_t$ and its associated semigroup $(P_t)_t$:

(1) *$(X_t)_t$ satisfies the Markov property;*
(2) *μ is its stationary measure, i.e., that $\int \mathbb{E}[f(X_t)|X_0 = x]\mu(\mathrm{d}x) = \int f \mu(\mathrm{d}x)$;*
(3) *$P_t f(x) = \sum_{S\subset\{1,\ldots,n\}} (1-e^{-t})^{|S|} e^{-t(n-|S|)} \int f(x_1, \ldots, x_n) \prod_{i\in S} \mu_i(\mathrm{d}x_i)$;*
(4) *Each $f : \{-1,1\}^n \to \mathbb{R}$ can be written as $f = \sum_{S\subset\{1,\ldots,n\}} \hat{f}(S) u_S$ for appropriate coefficients $\hat{f}(S)$ and $u_S(x) = \prod_{i\in S} x_i$*[2]*. Recall that using this representation, you defined in class an operator $T_{e^{-t}} f(x) = \sum_{S\subset\{1,\ldots,n\}} e^{-t|S|}\hat{f}(S) u_S$ for any $t \geq 0$. Show that $T_{e^{-t}} = P_t$.*
(5)

$$\mathcal{L}f := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0+} P_t f(x) := \lim_{t\downarrow 0} \frac{P_t f - f}{t} = -\sum_{i=1}^n \delta_i f,$$

where $\delta_i f(x) = f(x) - \int f(x_1, \ldots, y_i, \ldots, x_n)\mu_i(\mathrm{d}y_i) = f(x) - \frac{f(x_1,\ldots,1,\ldots,x_n) + f(x_1,\ldots,-1,\ldots,x_n)}{2}$. Note that $\delta_i\delta_i = \delta_i$ and $\delta_i\delta_j = \delta_j\delta_i$, hence, $-\sum_i \delta_i$ is a discrete Laplacian. Thus, let us write Δ instead of $\mathcal{L}$.

(6) *Conclude that $\frac{\mathrm{d}}{\mathrm{d}t} P_t f = \Delta P_t f$;*
(7) *$\delta_i P_t f = P_t \delta_i f$.*

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[1] Poisson process $(N_t)_{t\geq 0}$ of intensity $\lambda > 0$ on $\mathbb{R}_+$ is the counting process that satisfies $N_0 = 0$, has independent increments and $N_t - N_s \sim \mathrm{Poisson}(\lambda(t-s))$ for any $t \geq s \geq 0$. Recall that such a process can be constructed, for instance, in the following way: let $(T_i)_i$ be i.i.d. exponential random variables of intensity λ, then $N_t := \#\{n \in \mathbb{N}_0 : \sum_{i=1}^n T_i \leq t\}$ is the Poisson process of intensity λ.

[2] this definition of u_S might differ from the one used in class by a factor of $(-1)^{|S|}$

$\star$ **For fun.**

Exercise 3 (Hypercontractivity and log-Sobolev on the hypercube)**.** *Let P_t, μ be as in the previous exercise. Prove that the following are equivalent:*

(1) (log-Sobolev) $\mathrm{Ent}_\mu[f^2] \leq \frac{1}{2}\mathbb{E}_\mu[\sum_{i=1}^n (2\delta_i f)^2]$

(2) (Hypercontractivity) $\|P_t f\|_{\mathrm{L}^{q(t)}(\mu)} \leq \|f\|_{\mathrm{L}^q(\mu)}$ *for all* $q \geq 1, t \geq 0$, $f \in \mathrm{L}^q(\mu)$ *and* $q(t) = 1 + (q-1)e^{2t}$.

To show the implication (1) $\Rightarrow$ (2) *you may proceed as follows:*

- *Reduce to the case* $f \geq 0$.
- *Consider* $\log \|P_t f\|_{\mathrm{L}^{q(t)}}$ *and compute its derivative.*
- *Show that the derivative is non-positive by reducing to the following problem: for a function* $g \geq 0$, $p = q(t)$ *and* $p' = q'(t)$,

$$\sum_{i=1}^n \left(\mathbb{E}\left[\left(g^{p/2} - g_{(i)}^{p/2} \right)^2 \right] - \frac{p^2}{p'} \mathbb{E}[g^{p-1}\delta_i g] \right) \leq 0. \tag{0.1}$$

  Hint: apply log-Sobolev in this step.
- *Prove* (0.1) *using* $\left(\frac{b^{p/2}-a^{p/2}}{b-a}\right)^2 \leq \frac{p^2}{4(p-1)} \frac{b^{p-1}-a^{p-1}}{b-a}$ *(check this) and the previous exercise.*

To show the converse implication you may want to re-use the above second step.