TOPICS IN PROBABILITY. PART III: PHASE TRANSITION

Exercise sheet 10: Monotonicity and sharp transitions

Exercise 1 (Why monotonicity assumption?).

Recall that from Margulis-Russo lemma you have concluded that $p \mapsto \mathbb{E}_p[f]$ is nondecreasing whenever f is monotone, i.e. $f(x) \leq f(y)$ whenever $x_i \leq y_i$ for all i. Conclude this result easier in the following way:

- *Let $p \leq p'$. Show that there exists a coupling (X, X') of $\mathbb{P}_p$ and $\mathbb{P}_{p'}$ such that $X_i \leq X'_i$ a.s. for every $i = 1, \ldots, n$. Conclude that $p \mapsto \mathbb{E}_p[f]$ is non-decreasing whenever f is monotone.*

When f is not monotone, the map $p \mapsto \mathbb{E}_p[f]$ can be essentially arbitrary, as the following exercise shows:

- *Given any continuous function $h : [0,1] \to [0,1]$, construct functions $f_n : \{0,1\}^n \to \{0,1\}$ so that $\mathbb{E}_p[f_n] \to h(p)$ as $n \to \infty$ for all $p \in (0,1)$.*

For this reason, there is no meaningful notion of a (sharp) transition in general, unless an assumption such as monotonicity is made.

Exercise 2 (Examples of Boolean functions: influence and presence or absence of a phase transition.).

Let $f_n : \{0,1\}^n \to \{0,1\}$ be one of the following Boolean functions:

1. *(Dictatorship: the first bit determines the outcome) $f_n^D(x_1, \ldots, x_n) = x_1$;*
2. *(Tribes) Partition $\{1, \ldots, n\}$ into subsequent blocks of length $\log_2(n) - \log_2(\log_2(n))$ with perhaps some leftovers. Define $f_n^T(x_1, \ldots, x_n) = \mathbf{1}_A$, where A is the event that at least one of the blocks consists of only 1's;*
3. *(Iterated 3-majority function) Let $k \in \mathbb{N}$, consider a rooted 3-ary tree (the root vertex has degree 3, leaves degree 1 and other vertices degree 4) of depth k (in particular, there are $n = 3^k$ leaves). To each leaf we assign 0 or 1, and apply the 3-majority function, i.e., $M(x_1, x_2, x_3) = \mathbf{1}_{\{\sum_i x_i > 3/2\}}$, to determine the values of the vertices at depth $k-1$. We iterate this procedure until reaching the root, and define $f_n(x_1, \ldots, x_n)$ to be the value at the root. Example: if $k = 2$, we start with $x = (0,1,1;1,0,0;0,1,0)$, at depth 1 we get $(1,0,0)$, and hence, $f_n(x) = 0$.*

For all these examples, verify that f_n is monotone, compute the ith influence (for any i) and check "by hand" whether or not there is a phase transition.