Algebraic curves
Solutions sheet 11

May 29, 2024

Exercise 1. Let F be a projective plane curve.

1. Let $P \in \mathbb{P}^2_k$. Show that P is a multiple point of F if, and only if, $F(P) = F_X(P) = F_Y(P) = F_Z(P) = 0$.
2. Suppose F is irreducible. Show that F has finitely many multiple points.
3. Suppose F is nonsingular. Show that F is irreducible.

Now assume that F is irreducible of degree n.

4. Show that F has at most $\frac{1}{2}n(n-1)$ multiple points. (Hint: combine Bezout's theorem with previous questions.)

Solution 1.

1. Wlog, let $P = [1 : 0 : 0]$. P is a simple point $\iff$ $F_*(Y,Z)$ has a simple point at $(0,0)$. Denote $F_1(Y,Z)$ its degree 1 part. Then $F_1(Y,Z) \neq 0 \iff (F_*)_Y(0,0) \neq 0$ or $(F_*)_Z(0,0) \neq 0$. Moreover, $(F_*)_Y = (F_Y)_*$, $(F_*)_Z = (F_Z)_*$. Note that by the formula
$$dF = XF_X + YF_Y + ZF_Z$$
$F(P) = 0$ and $F_Y(P) = F_Z(P) = 0$ together with $P \in \{X \neq 0\}$ implies $F_X(P) = 0$.
2. Suppose $F \neq 0$. F and $F_\bullet$ share an irreducible component iff $F_\bullet = 0$ for degree reason ($F_\bullet$ denotes any partial derivative). As F is non zero, the formula above implies that one of the partial derivative is non zero. Then F has finitely many multiple points.
3. Suppose F reducible. $F = F_1F_2$. Then by Bezout $V(F_1)$ and $V(F_2)$ intersects at least in one point P, and F can not be smooth at P.
4. Wlog assume $F_X \neq 0$. Let P be a multiple point of F. Wlog, restrict to 2 cases :
   - $P = [1 : 0 : 0]$. As $P \in F$, we can write $F = X^{n-1}F_1(Y,Z) + \cdots + F_n(Y,Z)$. Then $F_X = (n-1)X^{n-2}F_1(Y,Z) + \cdots + F_{n-1}(Y,Z)$. then either $m_P(F_X) = m_P(F)$ if *char* $k \nmid n-1$, either $m_P(F_X) \geq m_P(F) + 1$ if *char* $k | n-1$.

- $P \in U_Y \cup U_Z$, wlog $P \in U_Y$. We write $P = [x_P : 1 : z_P]$. Then :
  $F_*(x_P + X, z_P + Z) = F_m(X, Z) + \dots$
  $(F_X)_*(x_P + X, z_P + Z) = (F_m)_X(X, Z) + \dots$
  so that in general, $m_P(F_X) \geq m_P(F) - 1$.

We use Bézout's theorem for F and F_X, hence

$$n(n-1) = \sum_P I(P, F \cap F_X) \geq \sum_P m_P(F) m_P(F_X)$$

Then $\sum_P m_P(F) m_P(F_X) \geq \sum_P m_P(F)(m_P(F) - 1) \geq 2\#\{\text{multiple points of F}\}$.

Exercise 2. Let F be an affine plane curve.

1. Show that a line L is tangent to F at P if, and only if, $I(P, F \cap L) > m_P(F)$. This justifies the definition of tangent lines for projective plane curves.

Now, let F be a projective plane curve and P a simple point on F.

2. Show that the tangent line to F at P has equation $F_X(P)X + F_Y(P)Y + F_Z(P)Z = 0$.

Solution 2.

1. From properties of intersection numbers, $I(P, F \cap L) \geq m_P(F) m_P(L) = m_P(F)$ with equality if the L is not a tangent line of F at P.

2. Wlog assume $P \in U_X$. $P = [1 : y_P : z_P]$. In U_X, the tangent line has equation

$$(F_*)_Y(P)(Y - y_P) + (F_*)_Z(P)(Z - Z_P) = 0$$

   And recall that $(F_*)_\bullet(P) = (F_\bullet)_*(P) = F_\bullet(P)$. We also have $F_X(P) + y_P F_Y(P) + z_P F_Z(P) = 0$ so the affine equation of the tangent line is given by $F_X(P) + F_Y(P)Y + F_X(P)Z = 0$. Its homogenisation gives the answer.

Exercise 3. Show that the following projective plane curves are irreducible; find their multiple points and the tangents at multiple points with their multiplicities:

1. $XY^4 + YZ^4 + XZ^4$
2. $X^2Y^3 + X^2Z^3 + Y^2Z^3$
3. $Y^2Z - X(X - Z)(X - \lambda Z),\ \lambda \in k$
4. $X^n + Y^n + Z^n,\ n > 0$

Solution 3.

1. $XY^4 + (X+Y)Z^4$ is irreducible since it is irreducible in $k(X,Y)[Z]$ (it is a UFD and $\frac{XY^4}{X+Y}$ is not a 4th power, and $X + Y$, XY^4 are coprime in $k[X,Y]$.

   Multiple points : $F_X = Y^4 + Z^4$, $F_Y = 4Y^3X + Z^4$, $F_Z = 4Z^3Y + 4Z^3X$.

$F(P) = F_X(P) = F_Y(P) = F_Z(P) = 0$ implies $yz^4 = 0$ and then $y = z = 0$ so $P = [1 : 0 : 0]$ is the only multiple point.

Tangent lines : $F_* = Y^4 + Z^4 + YZ^4$. $Y^4 + Z^4 = (Y + \zeta Z)(Y - \zeta Z)(Y + i\zeta Z)(Y - i\zeta Z)$. with ζ primitive 8th root of unity. Hence F has 4 tangent lines at P.

2. $F = X^2(Y^3 + Z^2) + Y^2Z^3$ is irreducible since it is irreducible in $k(Z,Y)[X]$ ($\frac{Z^3Y^2}{Z^3+Y^3}$ is not a square, and $Z^3 + Y^3$, Z^3Y^2 are coprime in $k[Y,Z]$.

   multiple point :

   - *char* $k = 2$: 3 multiple points $[1 : 0 : 0]$ (double point with double tangent line $X + Y$), $[0 : 1 : 0]$ (triple point with 3 tangents $Y - j^n\zeta Z$, ζ primitive 6th root) and $[0 : 0 : 1]$ (double point with a double tangent line X).
   - *char* $k = 3$: 3 multiple points $[1 : 0 : 0]$ (triple point with triple tangent line $X + Y$), $[0 : 1 : 0]$ (triple point with 3 tangents $Y - j^n\zeta Z$, ζ primitive 6th root) and $[0 : 0 : 1]$ (double point with 2 tangent liness $X \pm iY$).
   - *char* $k \neq 2, 3$: same multiple points, distinct tangent lines.

3. $F = Y^2Z - X(X - Z)(X - \lambda Z)$ irreducible since irreducible in $k(X,Z)[Y]$ and Z, $X(X - Z)(X - \lambda Z)$ coprime in $k[X,Z]$.

   - *char* $k \neq 2$: $[0 : 0 : 1]$ double point, with tangent lines $Y + iX$, $Y - iX$ and $[1 : 0 : 1]$ double point with tangent lines $Y - X, Y + X$.
   - *char* $k = 2$. $P = [0 : 0 : 1]$ double point with double tangent line $Y + X$.

     $P = [x : y : 1]$ double point with double tangent line Y.

4. $X^n + Y^n + Z^n$, $n > 0$. Irreducible by Gauss's lemma.

   In *char* $k \nmid n$ no multiple points.

   char $k = p$, $n = p^r q$, $p \nmid q$. $X^n + Y^n + Z^n = (X^q + Y^q + Z^q)^{p^r}$. It gives directly the tangent line, with multiplicity p^r

Exercise 4. Find the intersection points and the intersection numbers of the following pairs of projective plane curves:

1. $Y^2Z - X(X - 2Z)(X + Z)$ and $Y^2 + X^2 - 2XZ$
2. $(X^2 + Y^2)Z + X^3 + Y^3$ and $X^3 + Y^3 - 2XYZ$
3. $Y^5 - X(Y^2 - XZ)^2$ and $Y^4 + Y^3Z - X^2Z^2$
4. $(X^2 + Y^2)^2 + 3X^2YZ - Y^3Z$ and $(X^2 + Y^2)^3 - 4X^2Y^2Z^2$

Solution 4.

1. Intersection : $P = [0 : 0 : 1]$. *char* $k \neq 2$: $I(P, F \cap G) = 2$ (look at deshomogenized poly, $\tilde{P}$ simple point of F_*, Y uniformizer of $\mathcal{O}_{\tilde{P}}(F_*)$.

   char $k = 2$: $I(P, F \cap G) = 6 = dim\ k[X,Y]/(X^3, Y^2 - X^2)$.

2. Intersection point : $[0:0:1]$, $I(P, F\cap G) = I((0,0), F_*\cap G_*) = 4$

   $[1:\zeta j^n:0]$, ζ primitive 6th root. Transversal intersection, $I=1$.

3. $degFdegG = 20$. Intersection points ($char\ k \neq 2$) :

   $[0:0:1]$, $I=9$,

   $[1:0:0]$, $I=9$,

   $[(1-u)^{-2} : 1 : u(1-u)^2]$, $I=1$ with $u^2-2u-1=0$.

4.