# Lecture 1:

- welcome & introduce
- go over what is on Moodle, ask if questions
- warn demanding class recommend to come & keep up
- exercise session also as office hours (recall ~~E~~ D as well)
- recommend Dimitri's student seminar on toric var.
- REVIEW! cannot do it in class

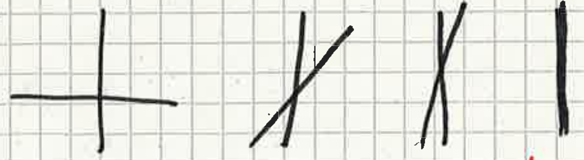
## Motivation:

Scheme theory is a powerful language to study algebraic geometry & related topics.
The main goal ~~of~~ the class is to develop this language.
Classical AG develops a correspondence between algebra & geometry. Still, it leaves grey areas:

- can we define varieties abstractly (not in $\mathbb{A}^n$ or $\mathbb{P}^n$)? (i.e., is there an equiv. of local charts?)
- what is a variety over $\Bbbk \neq \Bbbk$,? ~~or over $\mathbb{Z}$~~?
- can we interpret $x_0^2+x_1^2+x_2^2=0$ over ~~$\mathbb{F}_p$ and $\mathbb{Q}$~~ $\mathbb{Z}$ to interpolate between $\mathbb{F}_p$ and $\mathbb{Q}$?

recall that ~~varieties~~ alg sets are classically defined by radical ideals and varieties by prime ideals

- can we make sense of $f^{-1}(0)$ for $f : \mathbb{A}^2_{x,y} \longrightarrow \mathbb{A}^1_t$, $(x,y) \longmapsto x^2y$ as an object of the theory? I.e., does $V(x^2y)$ make sense as an object distinct from $V(xy)$?
- If we consider $V(x(x+ty)) \subseteq \mathbb{A}^2_{x,y}$ as $t$ varies, what is the limit as $t \mapsto 0$?



two lines?
V(x) is one line

The theory of sch[illegible]ws to address these questions in a sa[illegible] way

## Sheaves:

We will study our spaces via the functions that are allowed on them. To this end, this week we'll develop a theory to treat functions on spaces

**Example:** ~~$k$ a field~~

$$\mathbb{P}^1_{\mathbb{C}} = \mathbb{A}^1_{\mathbb{C}} \cup \mathbb{A}^1_{\mathbb{C}}$$

where on one $\mathbb{A}^1$ we have variable $x_1$, on one $x_2$ and we identify

$$\mathbb{A}^1 \setminus \{0\} \longrightarrow \mathbb{A}^1 \setminus \{0\}$$
$$x_1 \longmapsto \frac{1}{x_2}$$

A regular function on $\mathbb{A}^1_{x_i}$ is in ~~[illegible]~~ $\mathbb{C}[x_i]$

If $p(x_1) = \sum_{i=0}^{n} a_i x_1^i$, then on $\mathbb{A}^1_{x_2}$ it is

$$p(x_2) = \sum_{i=0}^{n} a_i x_2^{-i}$$

So for it to be a regular function, we need $a_i = 0$ for $i>0$

So, the only global functions on $\mathbb{P}^1_{\mathbb{C}}$ are constant

This is true more generally for projective varieties over $k=\overline{k}$

It can be thought of as an algebraic version of the maximum modulus principle in analysis

So we need a tool to talk about functions on open subsets, as global functions are scarce.

**Def.:** Let $X$ be a topological space. A presheaf $\mathcal{F}$ of abelian groups consists of the data

(a) for every open $U \subseteq X$, an abelian group $\mathcal{F}(U)$

(b) for every inclusion $V \subseteq U$ of open sets, a morphism of ab groups $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$

such that

(1) for every $U$, $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$

(2) for every $W \subseteq V \subseteq U$, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

**Notation:**
- An element $s \in \mathcal{F}(U)$ is a section of $\mathcal{F}$ over $U$; sometimes we write ~~$\ast$~~ $\Gamma(U, \mathcal{F})$ for $\mathcal{F}(U)$;
- $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is called restriction map, and we may write $s|_V$ ~~$\ast$~~ instead of $\rho_{UV}(s)$

Think of restricting a $\mathcal{C}^0$ function to a smaller open, that is the intuition

**Remark:**
- a presheaf is a contravariant functor from the category of open sets on $X$ (objects the open sets, arrows the inclusions) to AbGrp
- the same def holds for rings, or any abelian category
- unlike Hartshorne, we did not ask $\mathcal{F}(\emptyset) = \{0\}$. This choice is necessary ~~for $\mathcal{F}$ to be a functor~~ to identify presheaves with functors

**Example:**
- $X$ topological space
  $\mathcal{C}^0(U) = \{ f : U \to \mathbb{R} \text{ continuous} \}$, with $\rho_{UV}$ restriction of functions
- $M$ $\mathcal{C}^\infty$-manifold, $\mathcal{C}^\infty(U) \overset{\text{def}}{=} \{ f : U \to \mathbb{R} \ \mathcal{C}^\infty \}$
- $X$ complex manifold, $\mathcal{O}_X(U) = \{ f : U \to \mathbb{C} \text{ holomorphic} \}$
- on $\mathbb{R}$, $\mathcal{B}(U) = \{ f : U \to \mathbb{R} \ \mathcal{C}^0 \text{ bounded} \}$

~~Obs~~ Presheaves need not be the usual functions we know, they can also be tools to detect the topology of $X$

- $X$ $\tau$ space, $G$ gp, ~~$\ast$~~ $U \mapsto G$ for all $U$
- $X$ topological space, $G$ abelian group w/ discrete topology, $\underline{G}_X(U) = \{ f : U \to G \text{ continuous} \}$
  ~~in this case, $\underline{G}_X(U)$~~

You will see in the worksheet this is connected to detecting # components of $U$

- $X = \mathbb{R}$ with Euclidean topology, $\mathcal{F}(\mathbb{R}) = \mathbb{Z}$, $\mathcal{F}(U) = \{0\}$ if $U \subsetneq \mathbb{R}$

~~Def.: A presheaf~~

**Want:** - check if two sections are the same by local inspection

- glue

**Def.:** A presheaf $\mathcal{F}$ on $X$ is called sheaf if the following conditions hold. Let $\{V_i\}_{i\in I}$ be an open covering of an open $U$. Then,

(1) (uniqueness) if $s, t \in \mathcal{F}(U)$ and $s|_{V_i} = t|_{V_i}$ for all $i$, then $s = t$; and

(2) (gluing) if we have elements $s_i \in \mathcal{F}(V_i)$ for each $i$ such that $s_i|_{V_i\cap V_j} = s_j|_{V_i \cap V_j}$ for every $i,j \in I$, then $\exists s \in \mathcal{F}(U)$ s.t. $s|_{V_i} = s_i$ for all $i$

**Note:** if $\mathcal{F}$ is a sheaf, the uniqueness forces $\mathcal{F}(\emptyset) = \{0\}$

The first 3 examples are sheaves as they match our expectations. The fifth is discussed in the worllsheet. Ask about 4 and 6

4 ↓ no gluing $\mathbb{R} = \bigcup_{i=1}^{\infty}(-i,i)$, $s_i = x|_{(-i,i)}$

6 → no uniqueness, the sections on $\mathbb{R}$ all restrict to $0$

**Example:** • $X$ ~~$=\{x,y\}$ with discrete~~ top. space, $G$ group

$\mathcal{F}(U) = G$ for all $U$, $r_{UV} = id_G$

in general, it is not a sheaf (more in worsheet

• e.g., $X = \{x,y\}$, discrete top., $G = \mathbb{Z}$

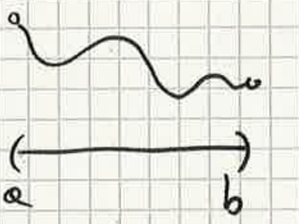
$$\begin{tikzcd}
 & \mathcal{F}(\{x,y\}) = \mathbb{Z} \arrow[dl] \arrow[dr] & \\
\mathcal{F}(\{x\}) = \mathbb{Z} \arrow[dr] & & \mathcal{F}(\{y\}) = \mathbb{Z} \arrow[dl] \\
 & \mathcal{F}(\emptyset) = \mathbb{Z} &
\end{tikzcd}$$

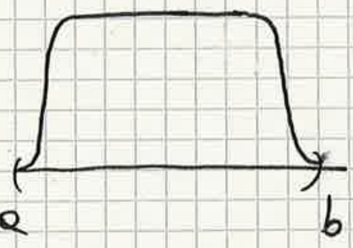
ask how to modify to get a sheaf ($\mathcal{F}(\emptyset) = \{0\}$, $\mathcal{F}(\{x,y\}) = \mathbb{Z}^{\oplus 2}$)

## Algebra interludium:

Why haven't we heard of sheaves earlier, in analysis or manifolds?
In a way one can get around the need of functions only defined on opens:

if $f \in \mathcal{C}^\infty((a,b))$, for any small $\varepsilon > 0$, we may find a bump function $g \in \mathcal{C}^\infty(\mathbb{R})$ s.t.
$g(x) = 1$ on $(a+\varepsilon, b-\varepsilon)$, $g(x) = 0$ if $x < a + \frac{\varepsilon}{2}$ or $x > b - \frac{\varepsilon}{2}$





so, $f \cdot g \in \mathcal{C}^\infty(\mathbb{R})$
and $f \cdot g|_{(a+\varepsilon, b-\varepsilon)} = f|_{(a+\varepsilon, b-\varepsilon)}$

These bump functions are not analytic, nor algebraic, so we cannot use them to extend a power series or a rational function.

So, we really need sheaves. And the following algebra tool will have us make sense of functions at a point

**Goal:** "functions at a point $p \in X$"

**Def.:** A directed set is a pair $(S, \leq)$, where $S$ is a set and $\leq$ is a transitive and reflexive relation on $S$ s.t. $\forall x, y \in S$, $\exists z \in S$ s.t. $x \leq z \wedge y \leq z$.

**Example:** $X$ topological space, $p \in X$ point
$S = \{U \subseteq X \text{ open} \mid p \in U\}$, $U \leq V$ if $V \subseteq U$
(⚠ inclusion reversing!)

**Def.:** A direct system $(A_*, f_{**})$ of abelian groups on a directed set $(I, \leq)$ is the datum of
(1) an ab grp $A_i$ for every $i$
(2) for every $i \leq j$, homomorphisms $f_{ij} : A_i \to A_j$ s.t.
 (i) $f_{ii} = id_{A_i}$
 (ii) $f_{jk} \circ f_{ij} = f_{ik}$ for every $i \leq j \leq k$

**Note:** if $(I, \leq)$ is thought as a category with objects $i$'s and arrows $i \to j$ if $i \leq j$, then $(A_*, f_{**})$ is a functor $(I, \leq) \longrightarrow AbGrp$

The intuition is that we want to follow the arrows in $I$ and get a limit object

**Def.:** Given $(A_*, f_{**})$ on $(I, \leq)$, we say $(B, g_*)$ is the **direct limit** if

(1) $B$ is an ab grp

(2) $\exists$ homomorphisms $g_i : A_i \longrightarrow B$ s.t. $g_j \circ f_{ij} = g_i$ for $i \leq j$

$$\begin{tikzcd}
A_i \arrow[rr, "f_{ij}"] \arrow[dr, "g_i"'] & & A_j \arrow[dl, "g_j"] \\
 & B &
\end{tikzcd}$$

and, given an abelian group $C$ and homomorphisms $h_i : A_i \longrightarrow C$ with $h_i = h_j \circ f_{ij}$ for $i \leq j$, the

(3) $\exists !$ homo $\alpha : B \longrightarrow C$ s.t. $\alpha \circ g_i = h_i$ for all $i$

$$\begin{tikzcd}
A_i \arrow[d, "f_{ij}"'] \arrow[dr, "g_i"] \arrow[drr, bend left, "h_i"] & & \\
 & B \arrow[r, "\alpha"] & C \\
A_j \arrow[ur, "g_j"'] \arrow[urr, bend right, "h_j"'] & &
\end{tikzcd}$$

By (3), if it exists, it is unique up to iso, and so we may write

$$\varinjlim_i A_i$$

**Prop.:** The direct limit $\varinjlim_i A_i$ of a dired system of ab grps exists and can be described in the following ways:

(1) let $G \subseteq \bigoplus_i A_i$ be the subgroup generated by the elements $x - f_{ij}(x)$, where $x \in A_i$, $j \geq i$

$$\begin{tikzcd}
A_i \arrow[r] \arrow[dr, "g_i"'] & \bigoplus_i A_i \arrow[d] \\
& \left(\bigoplus_i A_i\right)/G = \varinjlim_i A_i
\end{tikzcd}$$

obvious composition

(2) consider $\coprod_i A_i / \sim$, where $\sim$ is the eq. rel. generated by $x \sim f_{ij}(x)$ for any $x \in A_i$, $i \leq j$

in the second case the quotient becomes a group as any two $x \in A_i$, $y \in A_j$ can be added in $A_k$, $i \leq k$, $j \leq k$ as $f_{ik}(x) + f_{jk}(y)$

**Proof:** exercise

Stalks & germs:

**Def.:** Let $\mathcal{F}$ be a presheaf on $X$. The stalk $\mathcal{F}_x$ of $\mathcal{F}$ at $x$ is the group

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U)$$

where the direct limit is taken over open nbh of $x$ via Pov.

**Rmk.:**
- By (2) in Prop., $\mathcal{F}_x = \{(U, s) \mid x \cancel{p} \in U \text{ open}, s \in \mathcal{F}(U)\} / \sim$ where $(U, \cancel{f} s) \sim (V, t)$ if $\exists\, x \in W \subset U \cap V$ s.t. $s|_W = t|_W$
- if $\mathcal{F}(U)$'s are rings, so is $\mathcal{F}_x$

An element of $\mathcal{F}_x$ is a germ

If $x \in U$, $s \in \mathcal{F}(U)$, $s$ ~~has a natural~~ induces a germ $s_x \in \mathcal{F}_x$ via $\mathcal{F}(U) \longrightarrow \mathcal{F}_x$ (morphism $g_i$ in univ prop)

Example: • $\mathbb{R}$ and $\mathcal{C}^0$

$s = x$ in $\mathcal{C}^0(\mathbb{R})$, $t$  in $\mathcal{C}^0(\mathbb{R})$

$s_0 = t_0$ in $\mathcal{C}^0_0$

• $X = \mathbb{C}$, $\mathcal{O}_\mathbb{C}$ holomorphic functions

consider $0 \in U$ open, $s, t \in \mathcal{O}_X(U)$ s.t. $s_0 = t_0$

Then $\exists\, 0 \in V \subseteq U$ s.t. $s|_V = t|_V$

$\Longrightarrow$ same holomorphic function on $V \underset{\text{uniqueness power series}}{\Longrightarrow} s = t$

lec 11

While stalks retain very local information with $\mathcal{C}^0$ or $\mathcal{C}^\infty$, it has more global info for holo or alg functions, since when those agree on an open, they need to agree

Lemma: $\mathcal{F}$ sheaf on $X$, $s, t \in \mathcal{F}(U)$. If $s_x = t_x$ for all $x \in U$, then $s = t$.

Proof: By ~~replacing~~ subtracting $t$, WMA $t = 0$

So, $s_x = 0_x$, so $\exists\, U_x \subseteq U$ s.t. $s|_{U_x} = 0$ and $x \in U_x$

As $U = \bigcup_{x \in U} U_x$, by uniqueness, $s = 0$

So, sheaves are preferrable as we can not only glue, but also work locally of stalks.

Also, sheaves form an abelian category, having kernels, coker, etc.

Morphisms:

Def.: Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on $X$. A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is the datum of morphisms $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for all $U$ s.t., if $V \subseteq U$ the diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\text{of } F \leftarrow \rho_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho_{UV} \rightarrow \text{of } \mathcal{G}"] \\
\mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{G}(V)
\end{tikzcd}$$

commutes

$\varphi$ is an isomorphism if $\exists \psi: \mathcal{G} \to \mathcal{F}$ s.t. $\varphi(U)$ and $\psi(U)$ are inverse for every $U$.

<u>Remark:</u> Given $\varphi: \mathcal{F} \to \mathcal{G}$, for every $p \in X$ we get
$\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ via the univ. prop. of $\varinjlim$

and, if $p \in U$,

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, "\varphi(U)"'] & \mathcal{F}_p \arrow[d, "\varphi_p"] \\ \mathcal{G}(U) \arrow[r] & \mathcal{G}_p \end{tikzcd}$$

commutes (i.e., $(\varphi(U)(s))_p = \varphi_p(s_p)$)

<u>Example:</u> $X = \mathbb{R}$, exp: $\mathcal{C}^0 \longrightarrow (\mathcal{C}^0)^*$ (nowhere vanishing)

↑with + ↑ with multiplication

With sheaves, one can check inj & surj on stalks

<u>Prop.:</u> $\varphi: \mathcal{F} \to \mathcal{G}$ morphism of sheaves. Then, $\varphi$ is an iso iff so is $\varphi_p$ for every $p \in X$.

<u>Proof:</u> $\Rightarrow$) immediate from univ prop of $\varinjlim$ (use $\varphi_p^{-1} := (\varphi^{-1})_p$)

$\Leftarrow$) Since $\varphi$ is a morphism of sheaves, if $\varphi(U)$ is invertible for all $U$, we may define $\varphi^{-1}$ via $\varphi^{-1}(U) = (\varphi(U))^{-1}$

ISTS $\varphi(U)$ bijective

① $\varphi_p$ injective for every $p \Longrightarrow \varphi(U)$ injective for every $U$

Fix $U$ consider $s \in \mathcal{F}(U)$ s.t. $\varphi(U)(s) = 0 \in \mathcal{G}(U)$
Then, $\forall p \in U$

$$0 = (\varphi(U)(s))_p = \varphi_p(s_p) \underset{\text{inj}}{\Longrightarrow} s_p = 0$$

$\mathcal{F}$ sheaf $\underset{\text{Lemma}}{\Longrightarrow} s = 0$

② $\varphi_p$ surjective for every $p$ and $\varphi(U)$ injective $\Longrightarrow \varphi(U)$ surj for every $U$

Fix $U$ and $t \in \mathcal{G}(U)$
$\forall p \in U$, $\exists s_p \in \mathcal{F}_p$ s.t. $\varphi_p(s_p) = t_p$
Fix such $s_p$. Then, $\exists p \in U_p \subseteq U$ s.t. $s_p$ is the germ of $s^{(p)} \in \mathcal{F}(U_p)$

By injectivity of $\varphi(U_p)$ and $\varphi_p(s_p) = t_p$, we have
$\varphi(U_p)(s^{(p)}) = t|_{U_p}$ (✱)

Fix $p, p'$. Then, as $\varphi(U_p \cap U_{p'})$ is injective and by (✱),
$s^{(p)}|_{U_p \cap U_{p'}} = s^{(p')}|_{U_p \cap U_{p'}}$

So, we may glue to obtain $s \in \mathcal{F}(U)$ s.t. $\varphi(U)(s) = t$.

## Kernels, etc.

Def.: If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, we define $\ker(\varphi)$ via $U \mapsto \ker(\varphi(U))$

Prop.: If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, so is $\ker(\varphi)$

Note: Given $\varphi : \mathcal{F} \to \mathcal{G}$ with $\mathcal{F}$ and $\mathcal{G}$ sheaves
$U \mapsto \text{im}(\varphi(U))$ and $U \mapsto \text{coker}(\varphi(U))$ ~~do not~~ in general are only presheaves

Example: $X = \mathbb{C}$, $\mathcal{F} = \mathcal{O}_{\mathbb{C}}^{hol}$ holo functions, $\mathcal{G} = (\mathcal{O}_{\mathbb{C}}^{hol})^*$ non-vanishing holo fun. with $\cdot$ as group law

$$\mathcal{F} \longrightarrow \mathcal{G} \quad \text{given by } f \mapsto \exp(f)$$

~~If $U \subseteq \mathbb{C}$ is simply connected, a holomorphic logarithm is defined. So,~~

If $U_p$ is a small disk centered at, we have a holo logarithm on $U_p$ (translate $\log(1+x) = \sum x^n$)
so $\exp(U_p)$ is ~~non bijective~~ [illegible] surjective

Yet, on $U = \mathbb{C} \setminus \{0\}$, there is no logarithm
so $z \in \mathcal{G}(\mathbb{C} \setminus \{0\})$ is not in the image of $\mathcal{F}(\mathbb{C} \setminus \{0\})$

In order to remedy this problem, we want a natural way to define a sheaf when given a presheaf

__Def.:__ If $\mathcal{F}$ is a presheaf on $X$, a morphism of presheaves $sh: \mathcal{F} \longrightarrow \mathcal{F}^+$ is a __sheafification__ of $\mathcal{F}$ if $\mathcal{F}^+$ is a sheaf and for any sheaf $\mathcal{G}$ and any morphism $f: \mathcal{F} \longrightarrow \mathcal{G}$, there exits unique $g: \mathcal{F}^+ \longrightarrow \mathcal{G}$ s.t.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "f"] \arrow[dr, "sh"'] & \mathcal{G} \\ & \mathcal{F}^+ \arrow[u, "g"'] \end{tikzcd}$$

commutes

~~Note:~~

__Prop.:__ The sheafification exists, it is unique up to iso, and $sh_x: \mathcal{F}_x \longrightarrow (\mathcal{F}^+)_x$ is an iso for all $x$.

__Note:__ sh is the left adjoint of the inclusion of sheaves into presheaves

__Proof:__
- uniqueness easy by universality ✓
- stalks in worksheet
- existence: sketch

given $U \subseteq X$ open, define

$$\mathcal{F}^+(U) := \left\{ f: U \longrightarrow \coprod_{p \in U} \mathcal{F}_p \text{ satisfying ① \& ②} \right\}$$

① $\forall x \in U, \ f(x) \in \mathcal{F}_x$ (send every pt to the corresp stalk)

② $\forall x \in U, \exists x \in V \subseteq U$ s.t. $\forall y \in V \ \ f(y) = s_y$
(on small enough nbh's it is already a section)
↳ think of lay on △

$sh: \mathcal{F} \longrightarrow \mathcal{F}^+$ given by $s \longmapsto s_x$

**Example:** • $X$, $G$

the sheafification of the "constant presheaf"
$U \mapsto G$ is the "constant sheaf"
$U \mapsto \{ f : U \to G \text{ continuous} \}$ where $G$ has disc $\tau$

• $X = \mathbb{R}$, $\mathcal{B}$ bounded $\mathcal{C}^0$ functions, then

$$\mathcal{B}^+ = \mathcal{C}^0$$

**Def.:** Given a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, we define $\operatorname{coker}(\varphi)$ as the sheafification of $U \mapsto \operatorname{coker}(\varphi(U))$
and $\operatorname{im}(\varphi)$ as " $U \mapsto \operatorname{im}(\varphi(U))$

**Def.:** $\varphi : \mathcal{F} \to \mathcal{G}$ morphism of sheaves is
- injective if $\ker(\varphi) = 0$
- surjective if $\operatorname{im}(\varphi) = \mathcal{G}$ (i.e. $\mathcal{G}$ is a sheafification for the presheaf $U \mapsto \operatorname{im}(\varphi(U))$)

**Note:** $\varphi$ is injective iff $\varphi(U)$ is injective for every $U$. Yet, $\varphi$ may be surjective while $\varphi(U)$ is not surjective. Surjectivity is only a local lifting of section, not necessarily global

**Def.:** A subsheaf $\mathcal{F}'$ of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ s.t. $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$ for every $U$ and the restrictions of $\mathcal{F}'$ are induced by $\mathcal{F}$.

**Example:** • $\varphi : \mathcal{F} \to \mathcal{G}$ morphism of sheaves, $\ker(\varphi)$ is a subsheaf

• $$0 \to \underline{\mathbb{Z}}_{\mathbb{C}} \xrightarrow{n \mapsto 2\pi i n} \mathcal{O}_{\mathbb{C}}^{hol} \xrightarrow[f \mapsto \exp(f)]{} \left(\mathcal{O}_{\mathbb{C}}^{hol}\right)^* \to 0$$

is a short exact sequence of sheaves

**Prop.:** Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.
TFAE:

(1) $\varphi$ is injective (i.e., $\ker(\varphi) = 0$)
(2) $\varphi(U)$ is " $\forall U$
(3) $\varphi_P$ " $\forall P$

(i) $\varphi$ is surjective (i.e., $\operatorname{im}(\varphi) = \mathcal{G}$)
(ii) $\operatorname{coker}(\varphi) = 0$
(iii) $\varphi_P$ surj $\forall P$

## Push-forwards etc.

If $\mathcal{F}$ is a sheaf on $X$ and $U \subseteq X$ is open, the restriction sheaf $\mathcal{F}|_U$ is given by $V \longmapsto \mathcal{F}|_U(V) := \mathcal{F}(V)$ for any open $V \subseteq U$.

**Def.:** Let $f : X \to Y$ be a continuous map

(1) for any sheaf $\mathcal{F}$ on $X$, we define the push-forward sheaf $f_*\mathcal{F}$ via

$$\cancel{\mathcal{F}_*} \; f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

This is in fact a sheaf

(2) for any sheaf $\mathcal{G}$ on $Y$, we define the inverse image sheaf $f^{-1}\mathcal{G}$ as the sheafification of the presheaf defined by

$$U \longmapsto \varinjlim_{V \supset f(U)} \mathcal{G}(V)$$

where the limit is taken over all the open sets containing $f(U)$

**Note:** if $\iota : Z \to Y$ is the inclusion of $Z$ in $Y$ with the induced topology, the inverse image sheaf $\iota^{-1}\mathcal{G}$ is also denoted by $\mathcal{G}|_Z$ and called restriction. If $Z$ is open in $Y$, this retrieves the previous notion. Note, $\forall P \in Z$, $(\mathcal{G}|_Z)_P = \mathcal{G}_P$.

**Example:** $X=\{x,y\}$ two points in $\mathbb{A}^1_k = Y$, $\iota : X \to Y$
$\mathbb{A}^1_k$ Zariski topology

Consider the constant sheaves $\underline{\mathbb{Z}}_X$ and $\underline{\mathbb{Z}}_Y$

Any open in $\mathbb{A}^1_k$ is connected, so $\underline{\mathbb{Z}}_Y(U) = \mathbb{Z}$ unless $U=\emptyset$

Let $V$ be an open in $\{x,y\}$, then

$$(\iota^{-1}\underline{\mathbb{Z}}_Y)^{\overset{pre}{\downarrow}}(V) = \begin{cases} \{0\} & \text{if } V=\emptyset \\ \mathbb{Z} & \text{otherwise} \end{cases}$$

before we sheafify

$\Longrightarrow$ $(\iota^{-1}\underline{\mathbb{Z}}_Y)^{pre}$ is the constant presheaf, and $\iota^{-1}\underline{\mathbb{Z}}_Y$ is the constant sheaf

Now, $U$ open in $X$

$$\iota_*\underline{\mathbb{Z}}_X(U) = \begin{cases} \{0\} & \text{if } U=\emptyset, \text{ as } \iota^{-1}(\emptyset)=\emptyset \\ \mathbb{Z} & \text{if } \#(U\cap X)=1 \\ \mathbb{Z}^2 & \text{if } X\subseteq U \end{cases}$$

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## Spectrum of a ring

Recall: if $k=\bar{k}$, the Nullstellensats gives

points $\xleftrightarrow{1:1}$ maximal ideals
irreducible closed $\xleftrightarrow{1:1}$ prime ideals
closed $\xleftrightarrow{1:1}$ radical ideals

~~So, we can think~~

So, we can think of the space as been made of max ideals.

Now, with schemes, we will make every prime a point

<u>Def.:</u> Let $R$ be a ring. We define its <u>spectrum</u> as

$$\mathrm{Spec}(R) := \{ \mathfrak{p} \subset R \mid \mathfrak{p} \text{ prime} \}$$

<u>Example:</u>

- $k$ $\quad \mathrm{Spec}(k) = \{ \overset{\text{maximal}}{(0)} \}$
- $k = \overline{k}$, $\mathrm{Spec}(k[x]) = \{(0)\} \cup \{ \overset{\text{max}}{(x-\lambda)} \mid \lambda \in k \}$ ✓ & othe
- $\mathrm{Spec}(\mathbb{R}[x]) \supseteq \{(0)\} \cup \{(x-\lambda) \mid \lambda \in \mathbb{R}\} \cup \{(x^2+\mu) \mid \mu > 0\}$ max
- $\mathrm{Spec}(\mathbb{Z}) = \{(0)\} \cup \underbrace{\{(2), (3), (5), \dots\}}_{\text{max}}$

We can provide it with a topology

<u>Def.:</u> We define a topology on $\mathrm{Spec}(R)$ by declaring that closed sets have the form

$$V(I) := \{ \mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq I \}$$

where $I \subseteq R$ is an ideal.

<u>Proposition:</u> (1) $I, J$ ideals of $R$, then $V(I) \cup V(J) = V(IJ)$

(2) $\{I_i\}_{i \in I}$ a set of ideals, then $\bigcap_i V(I_i) = V(\sum_i I_i)$

(3) (Nullstellensatz) $I, J$ ideals, then $V(I) \subset V(J)$ if and only if $\sqrt{J} \subseteq \sqrt{I}$.

<u>Proof:</u> (1) $IJ \subseteq I, J$ always, so $V(I) \cup V(J) \subseteq V(IJ)$

if $\mathfrak{p} \supseteq IJ$ and say $\mathfrak{p} \not\supseteq I$, then $\exists a \in I$ s.t. $a \notin \mathfrak{p}$ and $\forall b \in J$, $ab \in \mathfrak{p} \Rightarrow \forall b \in J$, $b \in \mathfrak{p} \Rightarrow J \subseteq \mathfrak{p}$

(2) $\sum_i I_i$ is the smallest ideal containing all $I_i$, so

$$\mathfrak{p} \supseteq \sum_i I_i \Leftrightarrow \forall i, \mathfrak{p} \supseteq I_i$$

(3) ~~Recall~~ Use that $\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \supseteq I \\ \text{prime}}} \mathfrak{p}$

~~So, $\sqrt{J} \subseteq \sqrt{I} \Leftarrow \bigcap$~~

<u>Note:</u> $V((0)) = \mathrm{Spec}(R)$, $V(R) = \emptyset$

<u>Corollary:</u> It is indeed a topology (1) + (2) + (note)

We call it Zariski topology

**Def.:** We define the principal open sets

$$D(f) := \mathrm{Spec}(R)\setminus V((f)) = \{\mathfrak{p}\in \mathrm{Spec}(R) \mid f\notin \mathfrak{p}\}$$

for any $f\in R$.

**Note:** These form a base of the topology
(if $I=(a_i \mid i\in I)$, $V(I)=\bigcap V(a_i)$ )

**Example:** • $\mathrm{Spec}\,\mathbb{Z}$, $I\subset \mathbb{Z}$ nontrivial ideal $\overset{PID}{\Longrightarrow} I=(f)$

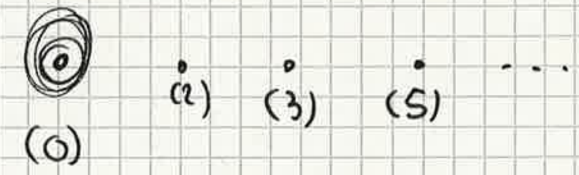
$$f=\prod_{i=1}^{m} p_i^{n_i}$$

So, $V(I)=\{\mathfrak{p} \mid \prod_{i=1}^{m} p_i^{n_i}\}=\{(p_1),\dots,(p_m)\}$

$\Longrightarrow$ closed sets are $\emptyset$, $\mathrm{Spec}\,\mathbb{Z}$, finite unions of primes

How about the point $(0)$? It is not closed
It is in the complement of every closed (except for $\mathrm{Spec}\,\mathbb{Z}$)
$\Longrightarrow$ $(0)$ belongs to every non-empty open
$\Longrightarrow$ $\overline{\{(0)\}}=\mathrm{Spec}\,\mathbb{Z}$

We have one dense point and closed points



• $\mathrm{Spec}(k[x])$ $k[x]$ again a PID
$I$ non-trivial ideal, then $I=(\prod_{i=1}^{m} f_i^{n_i})$, $f_i$ prime
$V(I)=\{(f_1),\dots,(f_m)\}$ closed pts
If $k=\bar{k}$, $f_i = x-\lambda_i$, $\lambda_i\in k$, so ~~these~~ correspond to the elements of $k$.

Over $\mathbb{R}$, also $(x^2+1)$ is a closed point, so we have more points

$(0)$ is again dense

**Observation:** Let $\mathfrak{p}, \mathfrak{q}$ be primes in $R$ with $\mathfrak{p} \subseteq \mathfrak{q}$. Let $I$ be an ideal of $R$

$$\mathfrak{p} \in V(I) \overset{\text{def}}{\Longleftrightarrow} \mathfrak{p} \supseteq I$$

So, if $\mathfrak{p} \in V(I) \Longrightarrow \mathfrak{q} \in V(I)$

$$\Longrightarrow \mathfrak{q} \in \overline{\{\mathfrak{p}\}}$$

Similarly, if $\mathfrak{r}$ is prime, $\mathfrak{r} \not\supseteq \mathfrak{p}$, then $\mathfrak{r} \notin V(\mathfrak{p})$ and $\mathfrak{r} \notin \overline{\{\mathfrak{p}\}}$

So, $\overline{\{\mathfrak{p}\}} = \{\mathfrak{q} \subseteq R \text{ prime} \mid \mathfrak{q} \supseteq \mathfrak{p}\}$

**Consequences:**
- closed points $\xleftrightarrow{1:1}$ maximal ideals
- if $A$ is an integral domain, $(0)$ prime and $\overline{\{(0)\}} = \operatorname{Spec} R$

**Example:**
- $\operatorname{Spec} \mathbb{K}[x,y]$, $\mathbb{K} = \overline{\mathbb{K}}$

  closed pts $\underset{\text{Nullst.}}{\longleftrightarrow}$ $(x-\lambda, y-\mu)$

  dense pt $\longleftrightarrow$ $(0)$

  neither closed nor dense $\longleftrightarrow$ ideals of height $1$

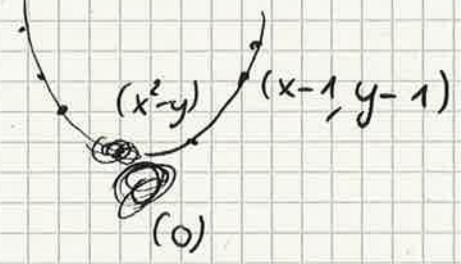
  e.g.: $(x^2-y) \supsetneq (0)$

  what is in its closure? Since $\dim R = 2$, we only need to add maximal ideals

  $(x-\lambda, y-\mu) \supseteq (x^2-y)$

  i.e., $x^2-y = f(x-\lambda) + g(y-\mu)$, so, if $x=\lambda$, $y=\mu$, then $x^2-y$ vanishes

  $\Rightarrow$ we add exactly the closed pts on the parabola!



- Given $R$ and $I$

$$\mathrm{Spec}(R/I) \xleftrightarrow{1:1} \{p \in \mathrm{Spec}\, R \mid p \supseteq I\} = V(I)$$

Also, given

$$\left\{\begin{matrix}\text{ideals in}\\ R/I\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{ideals in } R\\ \text{containing } I\end{matrix}\right\}$$

the correspondence is a homeomorphism between $\mathrm{Spec}(R/I)$ and $V(I)$ with the induced $\tau$ from $\mathrm{Spec}\, R$

e.g.: $\mathrm{Spec}\left(\Bbbk[x,y]/(x^2-y)\right) \underset{\text{homeo}}{\cong} V((x^2-y))$ in $\mathrm{Spec}(\Bbbk$

- by the Nullstellensatz, $V(I) = V(J) \Longleftrightarrow \sqrt{I} = \sqrt{J}$

so, by the previous point, if $\sqrt{I} = \sqrt{J}$

$$\mathrm{Spec}(R/I) \underset{\text{homeo}}{\cong} \mathrm{Spec}(R/J)$$

e.g.: $\mathrm{Spec}\left(\Bbbk[x,y]/(x^2)\right)$ and $\mathrm{Spec}\left(\Bbbk[x,y]/(x)\right)$

Ask how comfortable with localization (& primary decomp. to a lesser extent)

- If $S$ is a multiplicative set in $R$

$$\left\{\text{ideals in } S^{-1}R\right\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{ideals in } R \\ \text{disjoint from } S\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{prime ideals in} \\ S^{-1}R\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{prime ideals in } R \\ \text{disjoint from } S\end{matrix}\right\}$$

If $f \in R$, ~~$R_f$~~ and $f$ is not nilpotent then $R_f$ is a ring, with $R_f \cong R[x]/_{(xf-1)}$

$S = \{f^n \mid n \geq 0\}$

$J$ nontrivial is disjoint from $S \Leftrightarrow f \notin J$

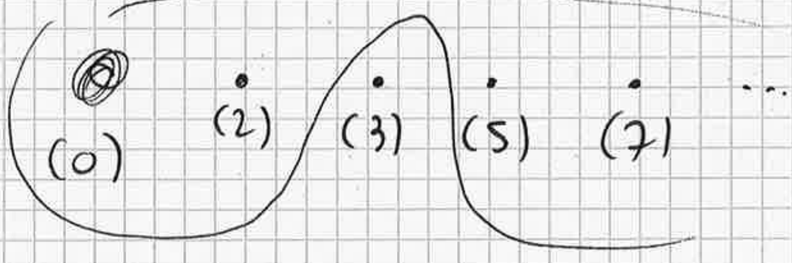
in particular, if $J = \mathfrak{p}$, $\mathfrak{p}$ is disjoint from $S \Leftrightarrow \mathfrak{p} \notin V(I)$

$$\Leftrightarrow \mathfrak{p} \in D(f)$$

So, $D(f)$ is homeomorphic to $\mathrm{Spec}(R_f)$

e.g.: $3 \in \mathbb{Z}$

$$\mathrm{Spec}(\mathbb{Z}_3) \underset{\text{homeo}}{\cong} \mathrm{Spec}(\mathbb{Z}) \setminus \{(3)\}$$



Now, we want to introduce a sheaf of functions on Spec(R). The sheaf together with the topological space, will recover the ring itself

**Want:**
- functions are rational expressions in $R$
  (e.g. $\frac{x^2+3}{x-2}$ on $\mathbb{A}^1$)
- $\mathcal{O}_{Spec(R)}(Spec(R)) = R$

**Idea:**

$$Spec(R) \longmapsto R$$

$$D(f) \longmapsto R_f$$

how about in general?

$$U \longmapsto S(U)^{-1}R$$

$$S(U) = \{ f \in R \mid f \notin p \ \forall p \in U \}$$

i.e. $U \subseteq D(f)$

we will interpret it as a nonvanishing condition of $f$ at $P$

**Note:** $f \notin p \iff f$ is a unit in the local ring $R_p$ (its max ideal is $p^e \subseteq R_p$)

$$\Updownarrow$$

$$0 \neq [f] \in \boxed{R_p / p^e} \to \text{field}$$

think of it as plugging in $p$ in $f$ to obtain a number

Restriction maps? If $V \subseteq U$, $S(U) \subseteq S(V)$
and we naturally have $S(U)^{-1}\{R\} \to S(V)^{-1}R$

There $\rho_{UV}$ satisfy the needed compatibilities by the properties of localization

Def.: $\mathcal{O}_{\text{Spec}(R)}$ is the sheafification of the presheaf just defined

Note: it is a sheaf of rings

**Lemma:** There is a natural isomorphism

$$\mathcal{O}_{\operatorname{Spec} R, p} \longrightarrow R_p$$

**Proof:**

$$\mathcal{O}_{\operatorname{Spec} R, p} \underset{\substack{\downarrow \\ \text{sheafification} \\ \text{preserves stalks}}}{\cong} \mathcal{O}^{pre}_{\operatorname{Spec} R, p} \underset{\substack{\downarrow \\ \text{def}}}{=} \varinjlim_{U \ni p} S^{-1}(U) R \underset{\substack{\downarrow \\ \text{characterization}}}{=} \{ (U, \tfrac{r}{f}) \mid p \in U, r \in R, f \in S(U) \} / \sim$$

Recall, $S(U) = \{ f \in R \mid f \notin q \text{ for all } q \text{ in } U \}$

If $p \in U$, then $S(U) \subseteq R \setminus p$, so by localization

$$S^{-1}(U) R \longrightarrow (R \setminus p)^{-1} R = R_p$$

By compatibility of localization

$$\varphi_p : \mathcal{O}_{\operatorname{Spec} R, p} = \varinjlim_{U \ni p} S^{-1}(U) R \longrightarrow R_p$$

SURJECTIVITY: fix $\frac{r}{f} \in R_p$, then $f \notin p \Longleftrightarrow p \notin V(f)$

$$\Updownarrow$$

$$p \in D(f)$$

So, $\frac{r}{f} \in S(D(f))^{-1} R$, $p \in D(f)$

and $\varphi_p : [(D(f), \frac{r}{f})] \longmapsto \frac{r}{f} \in R_p$

INJECTIVITY: Let $x \in \ker(\varphi_p)$. We may represent $x$ by $(U, \frac{r}{f})$ for some $U \ni p$, $f \in S(U)$, $r \in R$

By def of $\varphi_p$, $\varphi_p(x) = \frac{r}{f}$ as a ratio in $R_p$

$\Longrightarrow \exists\, h \in R \setminus p$, $hr = 0 \Longrightarrow \frac{r}{f} = 0$ in $S(D(h))^{-1} R$ (note, $h \in S(D(h))$ )

Since $h \in R \setminus p \Leftrightarrow h \notin p \Leftrightarrow p \in D(h)$

So, $\left(D(h), \frac{r}{f}\right)$ is a representative of $x$ as well

$\Rightarrow \quad x = 0 \in \mathcal{O}_{\operatorname{Spec} R, \mathfrak{p}}$

Corollary: $\mathcal{O}_{\operatorname{Spec} R}$ is the sheaf

$$U \longmapsto \left\{ s : U \longrightarrow \coprod_{\mathfrak{p} \in U} \,\middle|\, s \text{ satisfies (1) \& (2)} \right\}$$

(1): $s(\mathfrak{p}) \in R_{\mathfrak{p}}$ for each $\mathfrak{p} \in U$

(2) ~~there exists V~~ for each $\mathfrak{p} \in U$, $\exists V \subseteq U$ open, $\mathfrak{p} \in V$ ~~s.th.~~ and $r, f \in R$ s.t. $\forall \mathfrak{q} \in V$, $s(\mathfrak{q}) = \frac{r}{f}$ in $R_{\mathfrak{q}}$ and $f \notin \mathfrak{q}$

Proof: Lemma + def sheafification

Now we understand stalks and germs. We also want to understand sections on distinguished opens

Recall: $D(f)$ form a basis!

Prop.: For every $f \in R$, there are natural isomorphisms

$$\varphi : R_f \longrightarrow \mathcal{O}_{\operatorname{Spec} R}(D(f))$$

In particular, $\Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) = R$.

Proof: For every presheaf, we have $\mathcal{F} \xrightarrow{sh} \mathcal{F}^+$, so for every $U$ $\mathcal{F}(U) \xrightarrow{sh(U)} \mathcal{F}^+(U)$

In our case, we have $sh(D(f)) : S^{-1}(D(f))R \longrightarrow \mathcal{O}_{\operatorname{Spec} R}(D(f))$

and $S(D(f)) = \{ f^n \mid n \geq 0 \}$, so
$S^{-1}(U)R = R_f$

no

~~INJECTIVITY~~: For simplicity, $f=1$, i.e., $R_f = R$, $D(f) = \operatorname{Spec} R$
Call $sh(\operatorname{Spec} R) = \varphi$

**INJECTIVITY:** Suppose $\varphi(r) = 0 \in \mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R)$

By Corollary, $\varphi(r)$ is the $0$-map $\operatorname{Spec} R \longrightarrow \coprod_{p \in \operatorname{Spec}(R)} R_p$
so $r=0$ in $R_p$ for all $p$.

So, $\forall p \in \operatorname{Spec}(R)$, $\exists\, a \in R \setminus p$, $ar = 0$

Consider the ideal $I = \operatorname{Ann}(r) = \{ b \in R \mid br = 0 \}$

Then, $\forall p$, $\exists\, a \in I$, $a \notin p$

$$\Rightarrow \forall p,\ I \not\subseteq p \Rightarrow V(I) = \emptyset = V((1))$$

$$\underset{\text{Nullst.}}{\Longrightarrow} 1 \in \sqrt{I} \underset{1^n=1}{\Longrightarrow} 1 \in I \Longrightarrow r = 1 \cdot r = 0$$

**SURJECTIVITY:** Fix $s \in \mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R)$

By Corollary, $\exists$ cover $\bigcup_i V_i = \operatorname{Spec}(R)$,
$a_i, g_i \in R$, $g_i \notin p$ for all $p \in V_i$, $s(p) = \frac{a_i}{g_i}$

In particular, by sheafification, $s|_{V_i} = \frac{a_i}{g_i}$. Also, $V_i \subseteq D(g_i)$

**Claim 1:** WMA $V_i = D(h_i)$ and $s|_{V_i} = \frac{a_i}{h_i}$

Proof of Claim 1: By basis, $V_i = \bigcup_j D(h_{ij})$

Then, $D(h_{ij}) \subseteq V_i \subseteq D(g_i)$, so

$$V((g_i)) \subseteq V((h_{ij})) \Rightarrow \sqrt{(h_{ij})} \subseteq \sqrt{(g_i)}$$

$$\Rightarrow h_{ij}^n \in (g_i) \text{ for some } n \text{, i.e., } h_{ij}^n = c g_i$$

$$\text{so } \frac{a_i}{g_i} = \frac{c a_i}{h_{ij}^n}$$

Since $D(h_{ij}) = D(h_{ij}^n)$, we may replace the $a_i$ by $ca_i$ and the $g_i$ by $h_{ij}^n$

**Claim 2:** WMA $|I| < +\infty$
(i.e., $\operatorname{Spec} R = \bigcup_{i=1}^{m} D(g_i)$ for some $m$)

**Proof of Claim 2:** ex 5 in sheet 2

for myself: $\operatorname{Spec} R = \bigcup_{i\in I} D(g_i)$, so

$$\emptyset = \bigcap_{i\in I} V(g_i) = V\Big(\sum_{i\in I} (g_i)\Big)$$

so $1 \in \sqrt{\sum_{i\in I}(g_i)} \Longrightarrow 1 \in \sum_{i\in I}(g_i)$

$\Rightarrow 1 = \sum_{j=1}^{m} c_j g_{i_j}$ for som $\{1,\dots,m\} \hookrightarrow I$

Consider $D(g_i)\cap D(g_j) = \{p \in \operatorname{Spec}R \mid g_i \notin p\} \cap \{p\in \operatorname{Spec}R \mid g_j \notin p\}$

$$\underset{\text{primality}}{=} \{p \in \operatorname{Spec}R \mid g_i g_j \notin p\} = D(g_i g_j)$$

general fact!

$$\frac{a_i}{g_i}\Big|_{D(g_ig_j)} = s\Big|_{D(g_ig_j)} = \frac{a_j}{g_j}\Big|_{D(g_ig_j)}$$ by sheaf restrictions

By the injectivity of $R_{g_ig_j} \longrightarrow \mathcal{O}_{\operatorname{Spec}R}(V(g_ig_j))$

$$\frac{a_i}{g_i} = \frac{a_j}{g_j} \quad \text{in } R_{g_ig_j}$$

$\Rightarrow \exists n \in \mathbb{N},\ (g_ig_j)^n (a_ig_j - a_jg_i) = 0$ (★)

As $|I| < +\infty$, we may pick $n$ independently of $i,j$

(★): $g_j^{n+1}(g_i^{n} a_i) - g_i^{n+1}(g_j^{n} a_j) = 0$

As $\bigcup_{i=1}^{m} D(g_i^{n+1}) = \operatorname{Spec} R$, $\bigcap_{i=1}^{m} V(g_i^{n+1}) = \emptyset$, so $1 \in \sum (g_i^{n+1})$

$$1 = \sum_{i=1}^{m} b_i g_i^{n+1}$$

$$r := \sum_{i=1}^{m} b_i g_i^n a_i \in R$$

$$\forall j,\quad g_j^{n+1} r = \sum_{i=1}^{m} b_i g_i^n a_i g_j^{n+1} \underset{\overset{\uparrow}{(\star)}}{=} \sum_{i=1}^{m} b_i g_i^{n+1} a_j g_j^n = a_j g_j^n \underbrace{\sum_{i=1}^{m} b_i g_i^{n+1}}_{1} = a_j g_j^n$$

So, $\forall j$, $g_j^n (r g_j - a_j) = 0$, so $\frac{r}{1} = \frac{a_j}{g_j}$ on $R_{g_j}$

$\Rightarrow$ $r = s|_{D(g_j)}$ for every $j$ $\Rightarrow$ $r \mapsto s$

---

## Ringed spaces

Def.: A ringed space $(X, \mathcal{O}_X)$ is a topological space $X$ with a sheaf of rings $\mathcal{O}_X$ ← the functions we allow on $X$

A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ with $f: X \to Y$ continuous and $f^\#: \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$ of sheaves of rings.

↓ $\mathcal{O}_X$ and $\mathcal{O}_Y$ live on different spaces; this wants to mean that, by precomposing with $f$, "allowed functions" on $Y$ become "allowed on $X$"

[redacted]

Non example: $(\mathbb{C}, \mathcal{O}_{\mathbb{C}}^{hol}) \longrightarrow (\mathbb{C}, \mathcal{C}^0)$ $f = id$

as we do not have $f^\#: \mathcal{C}^0 \longrightarrow f_* \mathcal{O}_{\mathbb{C}}^{hol} = \mathcal{O}_{\mathbb{C}}^{hol}$

Example: $(X, \mathcal{O}_X^{hol}), (Y, \mathcal{O}_Y^{hol})$ holomorphic cplx mfd, $f: X \to Y$ holomorphic, $f^\#$ given by $\mathcal{O}_Y^{hol}(U) \longrightarrow \mathcal{O}_X^{hol}(f^{-1}(U))$

$$s \longmapsto s \circ f$$

**Def.:** $(X, \mathcal{O}_X)$ is a <u>locally ringed space</u> if $\mathcal{O}_{X,x}$ is a local ring ($\exists!$ max ideal) $\forall x \in X$

**Example:** $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$

~~$(X, \mathcal{O}_X^{hol})$ cplx mfd~~ $(\mathbb{C}, \mathcal{O}_{\mathbb{C}}^{hol})$

**Non-example:** $R$ ring, not local (e.g. $\mathbb{Z}$)
$X$ $\tau$ space
$(X, \underline{R}_X)$

What are morphisms of locally ringed spaces?

**Note:** given $(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$

we have $f^{\#} : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$

$$\Big\updownarrow \ 1:1 \quad \text{ex 3 sh 2}$$

$$f^{\#} : f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$$

$$\Big\downarrow$$

$$\forall x \in X, \ (f^{-1}\mathcal{O}_Y)_x \xrightarrow{f_x^{\#}} \mathcal{O}_{X,x}$$

$$\Big\downarrow \quad \text{ex 3 sh 2}$$

$$\forall x \in X, \ f_x^{\#} : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

So whenever we have a map of ringed spaces, we have a map between the stalks

**Def.:** $(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ morphism of ringed spaces between locally ringed spaces is a morphism of locally ringed spaces if every

$$f_x^{\#} : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is a local homomorphism of local rings

i.e., $f^\#_x(\mathfrak{m}_{f(x)}) \subseteq \mathfrak{m}_x$, or, equivalently,

$$(f^\#_x)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$$

What does it mean geometrically?

~~Given any $V \ni f(x)$ open and $f \in \mathcal{O}_Y(V)$~~

Given any $V \ni f(x)$ open and $s \in \mathcal{O}_Y(V)$

$$\begin{tikzcd}
\mathcal{O}_Y(V) \arrow[r, "f^\#(V)"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{O}_X(f^{-1}(V)) \arrow[d] \\
\mathcal{O}_{Y,f(x)} \arrow[r, "f^\#_x"] \arrow[d] & \mathcal{O}_{X,x} \arrow[d, "\text{local}"] \\
\mathcal{O}_{Y,f(x)}/\mathfrak{m}_{f(x)} = k(f(x)) \arrow[r] & k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x \quad \text{field}
\end{tikzcd}$$

if morphism of ringed spaces

if locally ringed spaces

if morphism of locally ringed spaces

For $s \in \mathcal{O}_Y(V)$ we think of $[s] \in k(f(x))$ as its value at $f(x) \in V$.

local condition $\rightsquigarrow$ it makes sense to define $[f^\#(s)]$ at $k(x)$

i.e., "evaluate & then pull back" = "pull back & then evaluate"

What are the morphisms of locally ringed spaces between spectra?

**Proposition:** There is an equivalence

$$\left\{ \begin{array}{l} \varphi : A \longrightarrow B \\ \text{ring homom} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} (f, f^{\#}) : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \longrightarrow (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \\ \text{hom of locally ringed spaces} \end{array} \right\}$$

**Rmk.:** we have Rings $\hookrightarrow$ locally ringed spaces

**Proof:** we have seen that spectra are locally ringed spaces

Firste, given $\varphi : A \to B$, we define

$$f : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$$
$$\mathfrak{p} \longmapsto \varphi^{-1}(\mathfrak{p})$$

it is well defined, as we know preimages of primes are prime

**Continuity:** ~~$f^{-1}(V(I))$~~ check on basis of closed

~~$f^{-1}(V(g))$~~

~~$A$~~

$$g \in A, \; f^{-1}(V(g)) = f^{-1}(\{\mathfrak{q} \subseteq A \mid g \in \mathfrak{q}\}) =$$
$$= \{\mathfrak{p} \subseteq B \mid g \in \varphi^{-1}(\mathfrak{p})\}$$
$$= \{\mathfrak{p} \subseteq B \mid \varphi(g) \in \mathfrak{p}\} = V(\varphi(g))$$

in general, $f^{-1}(V(I)) = V(\varphi(I))$

__$f^{\#}$:__ we construct $\mathcal{O}^{pre}_{\operatorname{Spec}A} \longrightarrow f_* \mathcal{O}^{pre}_{\operatorname{Spec}B}$ and then sheafify (use $(f_*\mathcal{F})^+ \cong f_*(\mathcal{F}^+)$ for RHS)

i.e., if $V \subseteq \operatorname{Spec}A$ open

$$S(V)^{-1}A \longrightarrow S(f^{-1}(V))^{-1}B$$
$$\frac{a}{g} \longmapsto \frac{\varphi(a)}{\varphi(g)}$$

well defined as, if $g \notin \mathfrak{q}$ for all $\mathfrak{q}$ in $V$, $\varphi(g) \notin \mathfrak{p}$ for all $\mathfrak{p} \in f^{-1}(V)$

(by previous computation, $f^{-1}(V(g)) = V(\varphi(g))$ )

By universal properties of sheaffication, limit, and localization

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\
\mathcal{O}^{pre}_{\operatorname{Spec}(A)}(V) = S(V)^{-1}(A) \arrow[r, "\varphi(V)"] \arrow[d] & S(f^{-1}(V))^{-1}B = \mathcal{O}^{pre}_{\operatorname{Spec}(B)}(f^{-1}(V)) \arrow[d] \\
\mathcal{O}_{\operatorname{Spec}(A)}(V) \arrow[r, "f^{\#}(V)"] \arrow[d] & \mathcal{O}_{\operatorname{Spec}(B)}(f^{-1}(V)) \\
A_{\varphi^{-1}(\mathfrak{p})} = \mathcal{O}^{pre}_{\operatorname{Spec}(A),\varphi^{-1}(\mathfrak{p})} \arrow[r, "\varphi_{\mathfrak{p}} = f^{\#}_{\mathfrak{p}}"] \arrow[d, dash] & B_{\mathfrak{p}} = \mathcal{O}^{pre}_{\operatorname{Spec}(B),\mathfrak{p}} \arrow[d, dash] \\
= \mathcal{O}_{\operatorname{Spec}(A),\varphi^{-1}(\mathfrak{p})} & = \mathcal{O}_{\operatorname{Spec}(B),\mathfrak{p}}
\end{tikzcd}$$

$\mathfrak{p}$ prime in $B$ s.t. $f(\mathfrak{p}) \in V$

__Local:__

__General fact:__ $\varphi : A \to B$ ring hom, $\mathfrak{q} = \varphi^{-1}(\mathfrak{p})$, $\mathfrak{p}$ prime
$\Rightarrow \varphi_{\mathfrak{p}} : A_{\mathfrak{q}} \to B_{\mathfrak{p}}$ local

~~Check: $\mathfrak{m}_{\mathfrak{q}} = \mathfrak{q}A_{\mathfrak{q}}$, $\mathfrak{m}_{\mathfrak{p}} = \mathfrak{p}B_{\mathfrak{p}}$ $\varphi_{\mathfrak{p}}($~~

Proof of fact: As before, since $q = \varphi^{-1}(p)$, then $A \setminus q = \varphi^{-1}(B \setminus p)$, so $\varphi(A \setminus q) \subseteq B \setminus p$

$$\Longrightarrow \quad \begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A_q \arrow[r, "\varphi_p"] & B_p \end{tikzcd} \quad \text{well defined}$$

we need $\varphi_p(m_q) \subseteq m_p$

$m_q = \alpha(q) \cdot A_q$, and $\varphi_p(A_q) \subseteq B_p$, and $m_p = \beta(p) B_p$

So, ISTS $\varphi(q) \subseteq p$, but this is by assumption

So, we have

$$\left\{ \begin{matrix} \varphi : A \to B \\ \text{ring hom} \end{matrix} \right\} \xrightarrow{F} \left\{ (f, f^\#) : \operatorname{Spec} B \to \operatorname{Spec} A \right\}$$

Clearly we have $\xleftarrow{G}$ by Proposition:

$$\begin{tikzcd} f^\#(\operatorname{Spec} A) : \mathcal{O}_{\operatorname{Spec}(A)}(\operatorname{Spec} A)) \arrow[r] \arrow[d, phantom, "\parallel"] & \mathcal{O}_{\operatorname{Spec}(B)}(\underbrace{f^{-1}(\operatorname{Spec}(A))}_{\operatorname{Spec}(B)}) \arrow[d, phantom, "\parallel"] \\ A & B \end{tikzcd}$$

We have $G \circ F = \mathrm{id}$, as

$$\begin{tikzcd} A = \mathcal{O}_{\operatorname{Spec}(A)}(\operatorname{Spec}(A))^{pre} \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}_A = sh_A"'] \arrow[dr, phantom, "\text{by } sh"] & \mathcal{O}_{\operatorname{Spec}(B)}(\operatorname{Spec}(B))^{pre} = B \arrow[d, "sh_B = \mathrm{id}_B"] \\ A = \mathcal{O}_{\operatorname{Spec}(A)}(\operatorname{Spec}(A)) \arrow[r, "f^\#(\operatorname{Spec}(A))"'] & \mathcal{O}_{\operatorname{Spec}(B)}(\operatorname{Spec}(B)) = B \end{tikzcd}$$

by def of $f^\#$

Prop

Prop

Why $F \circ G = id$ ?

Start with $(f, f^{\#}) : (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}\,B}) \longrightarrow (\mathrm{Spec}\,A, \mathcal{O}_{\mathrm{Spec}\,A})$

we get $\varphi = f^{\#}(\mathrm{Spec}(A))$

Check: $\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}(B)$

This follows by the local condition:

$$\begin{tikzcd} A \arrow[r, "f^{\#}(\mathrm{Spec}A)"] \arrow[d] & B \arrow[d] \\ A_{f(\mathfrak{p})} \arrow[r, "f^{\#}_{\mathfrak{p}} = \varphi_{\mathfrak{p}}"] & B_{\mathfrak{p}} \end{tikzcd}$$

by assumption

$\varphi_{\mathfrak{p}}^{-1}(\mathfrak{m}_{B_{\mathfrak{p}}}) = \mathfrak{m}_{A_{f(\mathfrak{p})}}$

and $\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$ follows

$\Gamma(D_f, \mathcal{O}_{\mathrm{Spec}\,R}) = R_f$

So, $f$ is induced by $\varphi$. Then, by the Proposition and the definition of the sheaf map induced by $\varphi$, $f^{\#}$ is the sheaf map induced by $\varphi$.

---

Importance of locality:

Example: $A$ DVR (e.g. $\mathbb{C}[x]_{(x)}$), $B = \mathrm{Frac}(A)$ (e.g. $\mathbb{C}(x)$)

$\varphi : A \longrightarrow B$ inclusion only map of rings

$$\mathrm{Spec}(B) = \{(0)\} \longrightarrow \mathrm{Spec}(A) = \{(0), (Q)\}$$

Two morphisms of ringed spaces:

$(0) \longmapsto (0)$, corresponding to $\varphi$

on the open $\{0\}$ is $B \xrightarrow{\prime} B$, on $\mathrm{Spec}(A)$ is $A \hookrightarrow B$

$(0) \longmapsto (Q)$

~~on both opens it is~~ with same $f^{\#}$ as above

not induced by $\varphi$, as $f((0)) \neq \varphi^{-1}((0))$

Definition: An affine scheme $(X, \mathcal{O}_X)$ is a locally ringed space that is isomorphic as locally ringed space to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R.

A scheme is a locally ringed space $(X, \mathcal{O}_X)$ s.t. $\exists$ open covering $\{U_i\}_{i\in I}$ s.t. $(U_i, \mathcal{O}_{U_i} := \mathcal{O}_X|_{U_i})$ is an affine scheme

Observation: • $(\operatorname{Spec} R_f, \mathcal{O}_{\operatorname{Spec} R_f}) \cong (D(f), \mathcal{O}_{\operatorname{Spec} R}|_{D(f)})$
RHS is an affine scheme

- unlike manifolds, local charts are in general not isomorphic to one another

A morphism of schemes is a morphism of locally ringed spaces.

Examples: • If $(X, \mathcal{O}_X)$ is a scheme, $U \subset X$ open $(U, \mathcal{O}_U := \mathcal{O}_X|_U)$ is a scheme

- R ring, $\operatorname{Spec}(R[x_1, \dots, x_n]) =: \mathbb{A}^n_R$, the affine n-space over R

**Side note:** it could be, $S, T \subseteq$ multiplicative sets ~~systems~~, $S \neq T$, $S^{-1}R \cong T^{-1}R$

The biggest ~~sy~~ multiplicative ~~system~~ set giving a localization is the saturation of the multiplicative set

$$\varphi : R \longrightarrow S^{-1}R$$

saturation of $S$ is $\varphi^{-1}((S^{-1}R)^{*})$

**e.g.:**

$$\mathbb{Z}_{(4)} = \mathbb{Z}_{(2)}$$

$$\mathbb{Z}_{(6)} = \{x \in \mathbb{Z} \setminus \{0\} \mid 2 \mid x \text{ or } 3 \mid x\}^{-1} \mathbb{Z}$$

Consistent with $V((f)) = V((f^2))$, $D(f) = D(f^2)$, $R_f = R_{f^2}$

We will not stress it when used, but it is useful

**Example:** ~~Spec $k[x]_{(x)}$ = Spec~~

- $\mathrm{Spec}(k[x]) \setminus \{\underset{\substack{\uparrow \\ \text{origin}}}{(x)}\} =$ ~~Spec($k[x]_x$~~ $D(x) \cong \mathrm{Spec}(k[x]_x) =$

  $= \mathrm{Spec}(k[x, x^{-1}])$

  is ~~affine,~~ as isomorphic to

  ~~$V(y_1 y_2) \subseteq \mathrm{Spec}(k[y_1, y_2])$ via~~

  $\mathrm{Spec}(k[y_1, y_2]/(y_1 y_2 - 1))$ via

$$\cancel{k[x, x^{-1}]} \ \cancel{\mathrm{Sp}} \ k[y_1, y_2]/(y_1 y_2 - 1) \longrightarrow k[x, x^{-1}]$$

$$y_1 \longmapsto x$$

$$y_2 \longmapsto x^{-1}$$

- $X = \mathrm{Spec}(k[x,y]/(x^2, xy))$ VS $\mathrm{Spec}(k[t]) = \mathbb{A}^1_k$

we have $k[x,y] \longrightarrow k[x,y]/(x^2, xy) \underset{\substack{\uparrow \\ (x) \supseteq (x^2, xy)}}{\longrightarrow} k[x,y]/(x) \cong k[t]$

so, we have

$$\mathbb{A}^1_k \xrightarrow{f} X \xrightarrow{g} \mathbb{A}^2_k$$

$g$ homeomorphism onto $V(x^2, xy)$

$f$ is homeomorphis, since $\mathfrak{p}$ in $k[x,y]$ contains $(x^2, xy)$ if and only if contains $(x) = \sqrt{(x^2, xy)}$

$f$ is not an isomorphism, since $k[x,y]/(x^2,xy) \longrightarrow k[x,y]/(x)$ is not

Notice, one has 0-divisors and nilpotents, the othe doesn't

Consider $y \in k[x,y]/(x^2,xy)$, $t \in k[t]$. Notice

$$\varphi : k[x,y]/(x^2,xy) \longrightarrow k[t]$$
$$x \longmapsto 0$$
$$y \longmapsto t$$

So, by properties of localization, we have

$$\begin{tikzcd}
 & \left(k[x,y]/(x^2,xy)\right)_y \arrow[r, "\tilde{\varphi}"] & k[t]_t \\
\text{commute} & \cong \ \text{iso} \arrow[l] & \\
 & k[x,y]_y/(x^2,xy)k[x,y]_y & \text{clearly iso} \\
y \text{ unit} & \cong \ \text{iso} \arrow[l] & \\
 & k[x,y]_y/(x)k[x,y]_y & \\
\text{commute} & \cong \ \text{iso} \arrow[l] & \\
 & \left(k[x,y]/(x)\right)_y \arrow[uuuuuur, bend right] &
\end{tikzcd}$$

$$\Rightarrow \tilde{\varphi} \text{ iso} \Longrightarrow \mathbb{A}^1_k \setminus \{\text{origin}\} = \operatorname{Spec}(k[t]_t) \cong D(t) \cong D(y)$$

$$\begin{tikzcd}
 & D(y) \arrow[d, equal] \\
 & X \setminus V((y)) \arrow[d, equal] \\
\text{if } \mathfrak{p} \text{ in } k[x,y] \text{ contains } (x^2,xy) \text{ and } (x), \text{ it is } (x,y) & \arrow[l] \\
 & X \setminus \{(x,y)\}
\end{tikzcd}$$

We think of $X$ as a copy of the $y$-axis with additional scheme structure at one point. ~~This structure can be thought as a~~ ~~[illegible]~~

The $y$-axis points vertically, just have $y$'s. At $(0,0)$ we have a first order in the $x$ too (killed at second order)

- In general, if $I \subseteq R$, $R \longrightarrow R/I$ gives $\operatorname{Spec} R/I \longrightarrow \operatorname{Spec} R$, mapping homeomorphically to $V(I)$

- pretending we knew how to take intersections:

$$\operatorname{Spec}\left(\mathbb{C}[x,y]/(y-x^2)\right) \cap \operatorname{Spec}\left(\mathbb{C}[x,y]/(y)\right) =$$

$$= \operatorname{Spec}\left(\mathbb{C}[x,y]/(y-x^2, y)\right) \cong \operatorname{Spec} \mathbb{C}[x]/(x^2)$$

using
$V(I)\cap V(J) = V(I+J)$

$$\operatorname{Spec}\left(\mathbb{C}[x,y]/(y-x^2)\right) \cap \operatorname{Spec}\left(\mathbb{C}[x,y]/(y-\varepsilon)\right) \qquad \varepsilon \neq 0$$

$$= \operatorname{Spec}\left(\mathbb{C}[x,y]/(y-x^2, y-\varepsilon)\right) =$$

$$= \operatorname{Spec}\left(\mathbb{C}[x,y]/(x^2-\varepsilon, y-\varepsilon)\right)$$

$$= \operatorname{Spec}\left(\mathbb{C}[x,y]/(x-\varepsilon, y-\varepsilon) \oplus \mathbb{C}[x,y]/(x+\varepsilon, y-\varepsilon)\right)$$

Both cases Spec of $\mathbb{C}$ v.s. of dim 2!

This is not a proof, but suggests schemes can capture intersections

- $p$ prime in $R$

$$\begin{tikzcd}
R \arrow[r] \arrow[dr] & R_p \arrow[r] & \overbrace{R_p/\mathfrak{m}_p}^{\overset{\kappa(p)}{\shortparallel}} := pR_p \cong \mathrm{Frac}(R/p) \\
& R/p \arrow[ur] &
\end{tikzcd}$$

residue field at $p$: the field in which functions take values at $p$

So, we have

$$\begin{tikzcd}
\underline{\mathrm{Spec}(\kappa(p))} \arrow[r] \arrow[dr] & \mathrm{Spec}\, R_p \arrow[r] & \mathrm{Spec}\, R \\
& \mathrm{Spec}\, R/p \arrow[ur] &
\end{tikzcd}$$

local spectrum: not an open in Spec(R), but can be thought as an infinitesimally small nbh of $p$ in Spec R (topological $\bigcap$ of all nbh)

it is not a point in Spec R, but can still be thought as $p$: ~~[illegible]~~ it maps to $p$ in Spec R (spec of field only 1 point)

<u>Note</u>: $V(p) = \overline{\{p\}}$, think of ~~[illegible]~~ as the generic point of $V(p)$

---

<u>Def.</u>: Let $S$ be a scheme. A scheme over $S$ is a scheme $X$ with a morphism $X \to S$. A morphism of schemes over $S$ is

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr] \arrow[rr, phantom, shift right=4, "\circlearrowright"] & & Y \arrow[dl] \\
& S &
\end{tikzcd}$$

where $f$ is a morphism of schemes.
i.e., we want to preserve the structure morphism over $S$
For this category, we write $Sch(S)$, or $Sch(R)$ if $S = \mathrm{Spec}(R)$.

<u>Example:</u> • if $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(R)$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] & \arrow[r, phantom, "\overset{1:1}{\longleftrightarrow}"] & & A & & B \arrow[ll, "\varphi"'] \\
& \mathrm{Spec}(R) & & & & & R \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

it means $\varphi$ is a morphism of $R$-algebras

∘

$$\begin{tikzcd} \operatorname{Spec}\mathbb{C} = \operatorname{Spec}(\mathbb{R}[x]/_{(x^2+1)}) \arrow[rr] \arrow[dr] & & \operatorname{Spec}(\mathbb{R}[x]/_{(x^2+1)}) = \operatorname{Spec}(\mathbb{C}) \arrow[dl] \\ & \operatorname{Spec}\mathbb{R} & \end{tikzcd}$$

corresponding to $x \longmapsto -x$ is a morphism of $\mathbb{R}$-schemes, not of $\mathbb{C}$-schemes.

Theorem: Let $k$ be an algebraically closed field. There is a natural fully faithful functor

$$t : Var(k) \longrightarrow Sch(k)$$

(i.e., $Hom_{Var(k)}(V, W) \longrightarrow Hom_{Sch(k)}(t(V), t(W))$ bijective)
For any variety $V$, $V$ is homeomorphic to the closed points of $t(V)$ and the sheaf of regular functions on $V$ is the restriction of $\mathcal{O}_{t(V)}$ to the subset of its closed points

Example: $k[x_1, \dots, x_n]$, $\mathfrak{p}$ prime determining $V$,
then $t(V) = (\operatorname{Spec}(k[x_1, \dots, x_n]/_I), \mathcal{O}_{Spec})$

Idea: Given a topological space $X$, let $t(X)$ be the set of irreducible closed in $X$.
If $Y \subseteq X$ closed, naturally $t(Y) \subseteq t(X)$.
We define the closed in $t(X)$ to be the $t(Y)$ as $Y \overset{d}{\subseteq} X$ varies. do example of planar curve $Y$
A continuous $f : X_1 \longrightarrow X_2$ naturally induces

$$t(f) : t(X_1) \longrightarrow t(X_2)$$

Also,
$$X \longrightarrow t(X) \quad \text{is a homeomorphism onto}$$
$$x \longmapsto \overline{\{x\}} \quad \text{its image}$$

Read Prop II 2.6 in Hartshorne for details

Note: if $V$ affine variety, its irreducible sets are the primes in $A(V)$

Consequences: • As $t$ is fully faithful, we can identify $\mathrm{Var}(\Bbbk)$ with the image of $t$

• we will introduce properties of schemes to understand what $\mathrm{im}(t)$ is

---

Since our geometric objects are shaped from algebra, many of their properties will come from there.

Dimension:

Def.: • Let $X$ be a $\tau$-space. Its dimension is

$$\dim(X) := \sup\{n \in \mathbb{N} \mid \exists\, Z_0 \subsetneq \cdots \subsetneq Z_n \text{ irreducible } \underset{\text{closed in } X}{}\} \in \mathbb{N} \cup \{\infty\}$$

$X$ is Noetherian if every descending chain of closed subsets stabilizes

• Let $R$ be a ring. Its (Krull) dimension is

$$\dim(R) := \sup\{n \in \mathbb{N} \mid \exists\, \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ primes}\} \in \mathbb{N} \cup \{\infty\}$$

$R$ is Noetherian if every ascending chain of ideals stabilizes

Lemma: $\dim(R) = \dim(\mathrm{Spec}(R))$

Proof: exercise, use: $V(I)$ irreducible $\Longleftrightarrow \sqrt{I}$ prime

Lemma: $R$ Noetherian $\Longrightarrow \mathrm{Spec}(R)$ Noetherian

Note: • $\exists$ Noetherian ring $R$ s.t. $\dim(R) = +\infty$ (Nagata's example) suitable localization of $\Bbbk[X_{\mathbb{N}}]$

• localization and quotient preserve Noetherianity

• $R$ local and Noetherian $\Longrightarrow \dim(R) < +\infty$

• $\exists\, R$ not Noetherian s.t. $\dim(R) < +\infty$

($\Bbbk[x_1, x_2, \ldots] / (x_1^2, x_2^2, \ldots)$, only prime is $(x_1, x_2, \ldots)$ and $(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \cdots$ is infinite)

**Example:** • $\dim(k) = 0$

• $R$ PID not field, $\dim(R) = 1$

e.g.: $\dim \mathbb{Z} = 1$, $\dim k[x] = 1$

• $\dim R$ is only a topological notion, it does not capture more of $\operatorname{Spec}(R)$.

Indeed, $V(I) = V(J) \iff \sqrt{I} = \sqrt{J} \iff \forall \mathfrak{p}$ prime in $R$ $\mathfrak{p} \supseteq I \iff \mathfrak{p} \supseteq J$

$$\iff \operatorname{Spec}(R/I) \underset{\text{homeo}}{\simeq} \operatorname{Spec}(R/J)$$

e.g.: $\dim\left(\operatorname{Spec} k[x,y]/(x^2, xy)\right) = \dim\left(\operatorname{Spec} k[x,y]/(x)\right)$

**Fact:** $k$ field, $R$ fin. gen. $k$-algebra that is an integral domain, then

$$\dim R = ~~\dim~~ \operatorname{trdeg}_k \operatorname{Frac}(R)$$

e.g.: $\dim \mathbb{A}^n_k = \dim(k[x_1, \dots, x_n]) = \operatorname{trdeg}_k k(x_1, \dots, x_n) = n$

very useful in geometric setups

**Def.:** $R$ ring, $\mathfrak{p}$ prime, its height is

$$\operatorname{ht}(\mathfrak{p}) := \sup\{n \in \mathbb{N} \mid \mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_n = \mathfrak{p} \text{ primes}\} \in \mathbb{N} \cup \{\infty\}$$

We can relate this notion to suitable codimensions

**Note:** • $\dim R = \sup_{\mathfrak{p} \in \operatorname{Spec} R} \operatorname{ht}(\mathfrak{p})$

• if $X$ is a T-space, $Z \subseteq$ closed irreducible, then

$$\operatorname{codim}_X(Z) := \sup\{n \in \mathbb{N} \mid \exists\, Z = Z_0 \subsetneq Z_1 \subsetneq \dots \subsetneq Z_n \text{ irr closed}\}$$

in general, $\operatorname{codim}_X(Z) + \dim(Z) \leq \dim(X)$

for some funny spaces we may not find a maximal chain of closed through a prescribed closed

• so, $\operatorname{codim}_{\operatorname{Spec}(R)} V(\mathfrak{p}) = \operatorname{ht}(\mathfrak{p})$

**Example:** $\Bbbk[x_1,\dots,x_n] \supseteq (x_i)$

then, $ht((x_i)) = 1$

$\geq 1$: $(0) \subsetneq (x_i)$, as $(0)$ prime

$\leq 1$: assume $(0) \subsetneq \mathfrak{p} \subsetneq (x_i)$

consider $f \in \mathfrak{p}$, $f \neq 0$. Then $x_i \mid f$; write $f = x_i^m g$ s.t. $g \notin (x_i)$ (i.e. $x_i \nmid g$)

As $x_i \notin \mathfrak{p}$, $x_i \cdot x_i^{m-1} g \in \mathfrak{p} \Rightarrow x_i^{m-1} g \in \mathfrak{p}$

iterate $\Rightarrow g \in \mathfrak{p} \subseteq (x_i)$, ↯

~~Remark~~ **Note:**
- $\deg_{x_i} f = \deg_{x_i} g + m$, one can use it as well
- similar arguments using divisibility still work in UFDs (more with Krull's Hauptidealsatz)

## Additivity of dim & ht

We would like our spectra to be nice topological spaces, where the codimension is the difference of dimensions

**Theorem:** $R$ f.g. $\Bbbk$-algebra, $R$ domain] ↘ so, coord ring of classic alg. var

i.e. quotient of $\Bbbk[x_1,\dots,x_n]$

$\mathfrak{p} \subseteq R$ prime. Then,

$$\dim(R) = \dim R/\mathfrak{p} + ht(\mathfrak{p})$$

**Note:** $\geq$ easy, as $\dim R/\mathfrak{p}$ is measured by $\mathfrak{q}_l \supsetneq \dots \supsetneq \mathfrak{q}_0 = \mathfrak{p}$, and $ht(\mathfrak{p})$ by $\mathfrak{r}_0 \subsetneq \dots \subsetneq \mathfrak{r}_j = \mathfrak{p}$

Combine the chains

$\leq$ hard: show can find maximizing chains for $R$ containing $\mathfrak{p}$

**Corollary:** $R$ domain, f.g. $\Bbbk$-algebra, $\mathfrak{p}$ prime

$$\dim \operatorname{Spec}(R) = \dim \operatorname{Spec}(R/\mathfrak{p}) + \operatorname{codim} V(\mathfrak{p})$$

$$= \dim V(\mathfrak{p}) + \operatorname{codim} V(\mathfrak{p}) = \dim(\operatorname{Spec} R/\mathfrak{p}) + \dim(\operatorname{Spec} R_{\mathfrak{p}})$$

homeo ↵

**Recall:** primes in $R_p$ $\underset{1:1}{\longleftrightarrow}$ primes disjoint from $R\setminus p$ $\longleftarrow$ primes $q$ s.t. $q \subseteq p$

↑ i.e., the ones we use for height

$$\begin{tikzcd} & R/p \arrow[dr] & \\ R \arrow[ur] \arrow[dr] & & k(p) \\ & R_p \arrow[ur] & \end{tikzcd}$$

$\dim R/p = \dim V(p) = \dim \operatorname{Spec} R/p$

$\dim R$ $\|$ $\dim \operatorname{Spec} R$

$\dim(k(p)) = 0$

$\dim R_p = \operatorname{ht} p = \operatorname{codim} V(p$

**Remark:** We need all hypotheses

e.g.: $A = k[x]_{(x)}$, $R = A[y] = (k[x]_{(x)})[y]$, $p = (xy-1)$ in $R$

↑ localization of f.g.

idea: we retain just the generic pt of hyperbola, ↑ lose all closed pts

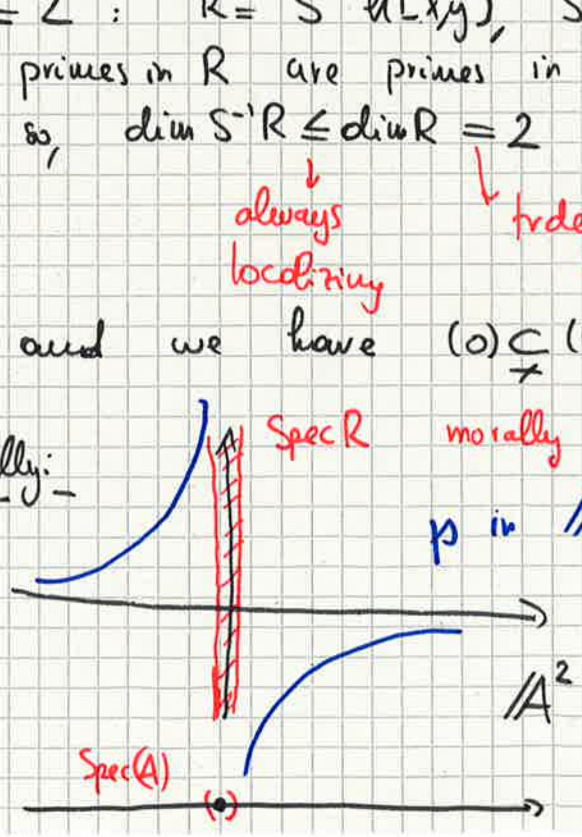
- $\operatorname{ht}(p) = 1$ by Krull's Hauptidealsatz (see later)
- $R/p = k[x]_{(x)}[y] \big/ (xy-1) = k[x]_{(x)}[x^{-1}] = k(x)$, so $\dim R/p = 0$
- $\dim R = 2$: $R = S^{-1}k[x,y]$, $S = k[x]\setminus(x)$

  primes in $R$ are primes in $k[x,y]$ disjoint from $S$

  so, $\dim S^{-1}R \underset{\text{always localizing}}{\le} \dim R \underset{\text{trdeg}}{=} 2$

  and we have $(0) \subsetneq (y) \subsetneq (x,y)$ in $k[x,y]\setminus S$

**Geometrically:**



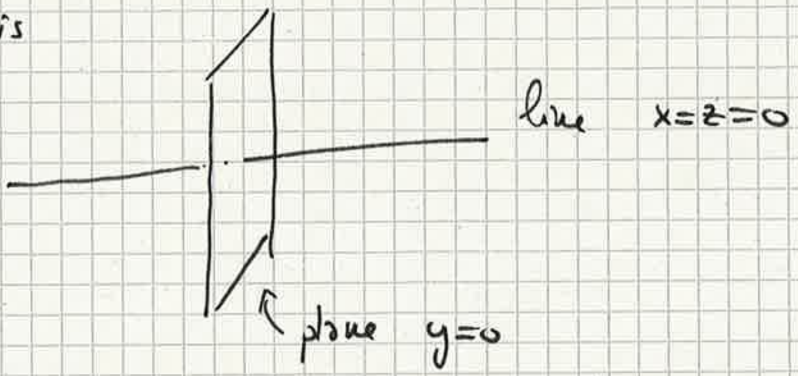
morally thin nbh of y-axis

$p$ in $\mathbb{A}^2$ is hyperbola

in $R$, geometrically we do not see the hyperbola and $\dim R/p = 0$

e.g.: $R = k[x,y,z]/(xy, yz)$ not a domain

its $\mathrm{Spec}(R)$ is



$\mathfrak{p} = (x, z)$

- $\dim R = 2$
- $\dim R/\mathfrak{p} = \dim k[z] = 1$
- $\mathrm{ht}(\mathfrak{p}) = 0$ as $R_\mathfrak{p} \cong k[x,y,z]_{(x,z)}/(xy,yz)^e \cong k(y)[x,z]_{(x,z)}/(x,z)^e \cong$

$$\cong k(y)[x,z]/[x,z] = k(y)$$

Geometrically: $R/\mathfrak{p}$ is the line, but $\mathrm{ht}(\mathfrak{p})$ computes its codimension in the line itself

Example: in $k[x_1, \dots, x_n]$, primes of height $k$ correspond to irreducible closed (classically, subvarieties) of codim $k$

This fully justifies our earlier example of $\mathbb{A}^2$, ht 1 being curves

Example: $R$ UFD, $f \in R$ prime. Then, $\mathfrak{p} = (f)$
$\mathrm{ht}\,\mathfrak{p} = 1$ (arguing as earlier in poly ring)
additivity $\Rightarrow$ $\dim V(f) = \dim R - 1$

So, we have an instance of: one eqt'n cuts dim by 1.
We'll see more with Krull's Hauptidealsatz

~~e.g.: recall Eisenstein's criterion: $A$ UFD, $p(x) = \sum_{i=0}^{n} a_i x^i \in A[x]$~~
~~if $\exists f \in A$ irreducible s.t.~~
~~$f \mid a_i$ for $0 \le i \le n-1$~~
~~$f \nmid a_n$~~
~~$f^2 \nmid a_0$ $\Longrightarrow$ $p$ irreducible~~

~~e.g.~~

**Def.:** An ideal $I \subseteq R$ is primary if, $\forall a,b \in R$
$ab \in I \Rightarrow a \in I$ or $b^n \in I$ for some $n \geq 1$
(i.e., $b \in \sqrt{I}$ )

**Note:**
- $\mathfrak{p}$ prime $\Longrightarrow$ $\mathfrak{p}$ primary
- $I$ primary $\Longrightarrow$ $\sqrt{I}$ prime
- $(x^2,xy) \subseteq \mathbb{R}[x,y]$, $\sqrt{(x^2,xy)} = (x)$ prime, yet $x \cdot y \in (x^2,xy)$, but $x \notin (x^2,xy)$ and $y^n \notin (x^2,xy)$ $\forall n \geq 1$ so not primary

**Theorem:** Let $R$ be a Noetherian ring, $I$ an ideal.
There exists a decomposition

$$I = \bigcap_{i=1}^{n} I_i$$

where each $I_i$ is primary and $\mathfrak{p}_i := \sqrt{I_i}$ are all distinct.
Furthermore, we may assume the minimality condition:

$$\text{for any } i = 1,\dots,n,\ I_i \not\supseteq \bigcap_{j\neq i} I_j$$

**Example:** $(x^2,xy) = (x) \cap (x^2,y)$

~~Recall:~~

**Note:** as $I \subseteq I_i$ $\forall i$, $V(I_i) \subseteq V(I)$ $\forall i$

$$\Rightarrow \bigcup_{i=1}^{n} V(I_i) \subseteq V(I)$$

yet, if $\mathfrak{p}$ is a prime, $\mathfrak{p} \in V(I)$, i.e.

$$\mathfrak{p} \supseteq I = \bigcap_{i=1}^{n} I_i \underset{\substack{\downarrow \\ \text{prime containing} \\ \text{finite } \cap \\ \text{contains one}}}{\Longrightarrow} \mathfrak{p} \supseteq I_{i_0} \text{ for some } i_0$$

$$\Rightarrow \mathfrak{p} \in V(I_{i_0})$$

$$\Rightarrow \bigcup_{i=1}^{n} V(I_i) \supseteq V(I)$$

$$\Rightarrow V(I) = \bigcup_{i=1}^{n} V(I_i)$$ union of irr. closed, as $\sqrt{I_i}$ prime

So, primary decomp gives us V(I) as union of irreducible closed. Yet, some are not needed set theoretically, they really only carry info about scheme structure

e.g.: $V((x^2, xy)) = V((x)) \cup \underbrace{V((x^2, y))} = V((x))$

this is important in the primary decomposition to detect the point where something extra is happening

We'll see now we get a 1-1 correspondence if I is radical

Def.: R ring, I ideal, a prime $\mathfrak{p}$ is a minimal prime of I if

- $\mathfrak{p} \supseteq I$
- $\mathfrak{p}$ is minimal among primes containing I

(equiv, $\operatorname{ht}(\mathfrak{p}/I) = 0$ in $R/I$)

Prop.: R Noetherian, $I = \sqrt{I}$, minimal primary dec. $I = \bigcap_{i=1}^{r} q_i$. Then, each $q_i$ is prime and $\{q_i\}_{i=1}^{r}$ coincides with the set of minimal primes of I (which, in particular, is finite)

Note: by ex. 3 sheet 3, $I = \sqrt{I} \iff (\operatorname{Spec} R/I, \mathcal{O})$ reduced

So, the proposition tells us how we can regard reduced spectra as finite unions

Geometrically: For any ideal, $V(I) = V(\sqrt{I})$ and we obtain a decomposition as finite union of irreducible $V(q_i)$, where

$$\bigcap_{i=1}^{r} q_i = \sqrt{I}$$

is a minimal primary decomposition.

Corollary: R Noetherian. The minimal primes of $I$ are exactly the prime ideals in a minimal primary decomp of $\sqrt{I}$. In particular, they are finitely many.

Example: $R = \mathbb{C}[x,y,z]$, $I = (xy, yz)$, $I = \sqrt{I}$

$$I = (y) \cap (x,z)$$

$$V(I) = V((y)) \cup V((x,z))$$

Why useful?

If R Noetherian, $\dim(R) < +\infty$, $I$ ideal

$$\dim R/I = \dim R/\sqrt{I} = \dim V(\sqrt{I}) =$$

$$= \max_{\substack{p \text{ minimal} \\ \text{of } I}} \dim V(p)$$

if R ℂ-alg f.t. domain

$$= \max_{\substack{p \text{ min} \\ \text{of } I}} \left( \dim A - ht(p) \right)$$

$$= \dim A - \min_{\substack{p \text{ min} \\ \text{of } I}} ht(p)$$

So, computing the dimension reduces to computing the height of minimal primes. Still, it can be hard!

Theorem (Krull's Hauptidealsatz): R Noetherian, $0 \neq r \in R$, $p$ minimal prime of $(r)$. Then

$$ht(p) \leq 1.$$

Corollary 1: If further R is a domain, $ht(p) = 1$

because $(0) \subsetneq (r) \subseteq p$

Corollary 2: If further R is a f.g. ℂ-algebra and domain, then $\dim R/(r) = \dim R - \min_{p \text{ min of } (r)} ht(p) = \dim R - 1$

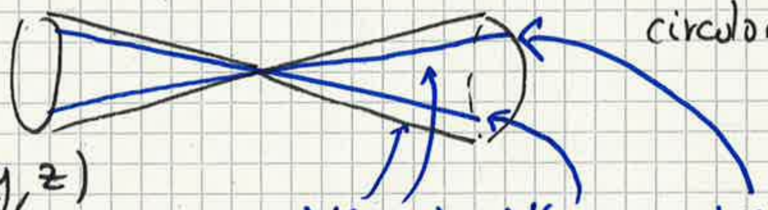
Slogan: one equation cuts the dimension by 1

Cor 2 geometrically: $V((r))$ is a union of hypersurfaces in $\operatorname{Spec}(R)$

e.g. $r = xy \in k[x,y]$

<u>Example:</u> • $R = k[x]/(x^2)$, $r = x$, only prime is $\mathfrak{p} = (x)$
$\operatorname{ht}\mathfrak{p} = 0$

• $R = k[x,y,z]/(xy-z^2)$, $\operatorname{Spec}(R)$ homeo to $V(xy-z^2) \subseteq \mathbb{A}^3_k$



$V((z)) = V((x,z)) \cup V((y,z))$

$z \in R$, $\underset{\substack{\uparrow \\ \text{not prime} \\ xy \in (z)}}{(z)} = (x,z) \cap (y,z)$

primes

R is not a UFD, so ~~[illegible]~~ primes of ht=1 are not necess. principal

• $k = \overline{k}$, $X = \overline{\{(t^2, t^3, t^4) \in \mathbb{A}^3_k \mid t \in k\}} \subseteq \mathbb{A}^3_k = \operatorname{Spec} k[x,y,z]$

surely need to add non-closed pts

expect $\dim X = 1$

• $\underset{\substack{\updownarrow \\ (x,y,z)}}{\{(0,0,0)\}} \subsetneq X \Rightarrow \dim X \geq 1$

• ~~$(t^4)^2 t$~~ $t^4 \cdot t^2 = (t^3)^2 \Rightarrow X \subseteq \underbrace{V((y^2 - xz))}_{\dim = 2}$

we need another equation

• $(t^2)^2 = t^4$, $\rightsquigarrow$ $X \subseteq V((x^2 - z))$

so, $X \subseteq V((y^2 - xz, x^2 - z))$

note, $\dim X \leq \dim V((y^2-xz, x^2-z)) = \dim k[x,y,z]/(y^2-xz, x^2-z)$

$$= \underbrace{\dim\left(k[x,y,z]/(y^2-xz)\right)}_{2} - \underbrace{\operatorname{ht}((x^2-z))}_{\substack{x^2 - z \neq 0 \text{ in ring, domain} \\ \text{so } \operatorname{ht} = 1}} = 1$$

**Claim:** $X = V((y^2 - xz, x^2 - z))$

check on closed points, enough

$(a,b,c) \in V((y^2 - xz, x^2 - z))$, so $c = a^2$, $b^2 = a^3$, i.e.,

$b = t^3$ for some $t$ s.t. $t^2 = a$. So, $(a,b,c) = (t^2, t^3, t^4)$

**Note:** $X$ is the image of $f : \mathbb{A}^1_k \longrightarrow \mathbb{A}^3_k$ given by

$$\begin{aligned} k[x,y,z] &\longrightarrow k[t] \\ x &\longmapsto t^2 \\ y &\longmapsto t^3 \\ z &\longmapsto t^4 \end{aligned}$$

so, as non-constant image of irreducible of dim 1, it has dim 1

~~Note: dim $X = 1$, but we need 2 equations~~

## Polynomial rings

**Theorem:** $R$ Noetherian ring, then

$$\dim R[x_1, \dots, x_n] = \dim R + n$$

**Example:**
- $\dim \mathbb{A}^n_k = n$ already know by trdeg
- $\dim \mathbb{A}^1_{\mathbb{Z}} = 2$

**Corollary:** $R$ Noetherian local ring, then

$$\dim R = \min\{n \in \mathbb{N}_{\geq 1} \mid \sqrt{\underbrace{(r_1, \dots, r_n)}} = \mathfrak{m}\}$$

any such collection realizing min is called system of parameters

slogan: dim is length ~~minimal~~ of SOP

(49)

## Height estimates

**Def.:** $I$ ideal, $\mathrm{ht}(I) := \inf\{\mathrm{ht}\,\mathfrak{p} \mid \mathfrak{p} \supseteq I \text{ prime}\}$

**Note:** can just restrict to minimal primes, which are finite if $R$ is Noetherian.

**Theorem:** $R$ Noetherian, then $\mathrm{ht}((r_1, \dots, r_s)) \leq s$.

**Note:** not immediate from HIS & induction, as a chain of primes computing ht of a minimal prime of $(r_1, \dots, r_s)$ may not contain the minimal primes of $(r_1, \dots, r_{s-1})$

**Geometric interpretation:** $R$ domain, $\Bbbk$-algebra f.g., then

$$\dim R/_{(r_1, \dots, r_s)} \geqslant \dim R - s$$

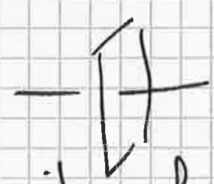
**Slogan:** $s$ equations cut dimension by at most $s$

**Def.:** An ideal $(r_1, \dots, r_s)$ s.t. $\mathrm{ht}((r_1, \dots, r_s)) = s$ is called set theoretic complete intersection

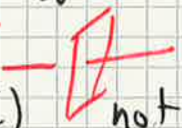
**Geometrically:** $V((r_1, \dots, r_s))$ is a set theoretic c.i. if it ~~can be cut out (set theoretically) by~~ has codimension $s$.

**Example:** • $V((x^2, xy)) \subseteq \mathbb{A}^2_\Bbbk$ is set th. c.i. as $\sqrt{(x^2, xy)} = (x)$

• $V((xy, yz)) \subseteq \mathbb{A}^3_\Bbbk$ not a set th c.i. as



$(xy, yz) = (y) \cap (x, z)$ it has $\mathrm{ht} = 1$ cannot be principal, as those give equidimensional loci

Remark: $R$ f.g. $\Bbbk$-algebra, domain
the height of $I$ and $\dim R/I$ capture the (co)dimension of the biggest irreducible component of $V(I)$ (the plane in  )

So keep in mind if $V(I)$ not equidimensional

$$V(I) = V(\sqrt{I}) \underset{\substack{\uparrow \\ \text{min} \\ \text{primary} \\ \text{dec} \\ \sqrt{I} = \bigcap_{i=1}^{n} \mathfrak{p}_i}}{=} \bigcup_{i=1}^{n} V(\mathfrak{p}_i)$$

$$\operatorname{ht}(I) = \operatorname{ht}(\sqrt{I}) = \min_{i=1}^{n} \operatorname{ht}(\mathfrak{p}_i) = \min_{i=1}^{n} \operatorname{codim} V(\mathfrak{p}_i)$$

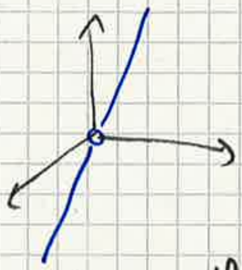
$$\dim R/I = \dim R - \min_{i=1}^{n} \operatorname{ht}(\mathfrak{p}_i) = \max_{i=1}^{n} \dim V(\mathfrak{p}_i)$$

## Projective schemes

Recall $\mathbb{P}^n_{\mathbb{C}} := \mathbb{C}^{n+1} \setminus \{(0,\dots,0)\} / \mathbb{C}^*$

where $\lambda \cdot (x_0,\dots,x_n) = (\lambda x_0,\dots,\lambda x_n)$

**Orbits:**
- origin
- lines through origin, with origin removed



So, $\mathbb{P}^n_{\mathbb{C}}$ parametrizes the lines through the origin in $\mathbb{A}^{n+1}_{\mathbb{C}}$

It is constructed as a group quotient. We will see how to do this algebraically

**Note:** $\dfrac{f}{g} \in \mathbb{C}(x_0,\dots,x_n)$ is $\mathbb{C}^*$-invariant iff $\deg(f)=\deg(g)$

- We will see how this will be a recurrent theme
- To define a value on $\mathbb{P}^n_{\mathbb{C}}$, a function needs to be constant along the lines (i.e., the orbits)

Def.: Let $(G,+)$ be an abelian monoid with neutral element $e$ (e.g., $\mathbb{N}$, $\mathbb{Z}$). A $G$-graded ring is a ring $S$ s.t.

$$S = \bigoplus_{g\in G} S_g$$

as an additive group such that

(1) $1 \in S_e$

(2) $\forall g,g' \in G, \quad S_g \cdot S_{g'} \subseteq S_{g+g'}$

~~For $g\in G$,~~

An element $s\in S$ is homogeneous if $s\in S_g$ for some $g$.

An ideal $I \subseteq S$ is homogeneous if

$$I = \bigoplus_{g\in G} (\underbrace{I\cap S_g}_{I_g}) \quad \text{(equiv., } I \text{ has a homogeneous generating set)}$$

If $S$ is an $\mathbb{N}$-graded ring, we call

$$S_+ := \bigoplus_{d>0} S_d \quad \text{the irrelevant ideal of } S$$

(note: it would not be an ideal in a $\mathbb{Z}$-graded ring)

Example: • $S = k[x_0, \dots, x_n]$ with degree grading

$S_+ = (x_0, \dots, x_n)$

$\operatorname{Spec} S = \mathbb{A}^{n+1}_k$, $\quad V(S_+) =$ origin

so, we identify the irrelevant ideal as something we do not want

• $V \subseteq \mathbb{P}^n_k$ classical proj var with ~~proj.~~ homog coordinate ring

$$S(V) = k[x_0, \dots, x_n] / \underbrace{I(V)}_{\text{homogeneous}}, \qquad S(V)_d = k[x_0, \dots, x_n]_d / I(V)_d$$

Goal: given an $\mathbb{N}$-graded ring $S$, construct a scheme $\operatorname{Proj}(S)$ s.t., if $S(V)$ is a homogeneous coordinate ring of a variety $V$,

$$\operatorname{Proj} S(V) = t(V)$$

Idea: $\mathbb{P}^n_k$ is the set of lines through $\underline{0}$ in $\mathbb{A}^{n+1}_k$. To define sets in $\mathbb{P}^n_k$, we define sets $C$ in $\mathbb{A}^{n+1}_k$ s.t., if $x \in C$, the line joining $\underline{0}$ and $x$ is in $C$.

$\Rightarrow$ need homogeneous polynomials!

e.g.: $x_0^2 + x_1^2 + x_2^2$ good: draw round cone and conic!

$x_0 + x_1^2$ bad: vanishes at $(-1, -1)$ not at $(-2, -2)$

~~Def. 1~~

Def.: Let $S$ be an $\mathbb{N}$-graded ring, we define

$$\operatorname{Proj} S := \{ \mathfrak{p} \subseteq S \mid \mathfrak{p} \text{ prime, homogeneous}, \ \mathfrak{p} \not\supseteq S_+ \}$$

think of rescaling

think of origin

Remark: • if $S_0 = k$ a field, $S_+$ is maximal (think of removing the origin in $\mathbb{A}^{n+1}_k$)

• if $S_0$ is not a field, there could be $\mathfrak{p} \in \operatorname{Proj} S$ s.t. $\mathfrak{p} \subseteq S_+$: $S = \mathbb{Z}[x]$, $(2) \in \operatorname{Proj}(S)$, $\underset{\overset{\parallel}{(0)}}{(2) \cap S_+} \neq \emptyset$

• $S_0$ is always a subring

**Def.:** If $I \subseteq S$ is a homogeneous ideal, we define

$$V(I) := \{ \mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \supseteq I \}$$

**Lemma:** $S$ $\mathbb{N}$-graded ring

homog!

(1) $I, J$ homogeneous ideals, then $V(I \overset{\cap}{\,} J) = V(I) \cup V(J)$

(2) $\{I_a\}_{a\in\Delta}$ collection of homogeneous ideals, then

$$V\Big(\sum_{a\in\Delta} I_a\Big) = \bigcap_{a\in\Delta} V(I_a)$$

(3) $V(S_+) = ~~\operatorname{Proj}~~\ \emptyset$, $\quad V((0)) = \operatorname{Proj} S$

Proof same as non-homogeneous case

**Corollary:** This defines a topology on $\operatorname{Proj}(S)$, called Zariski topology.

**Remark:** If $S$ is an $\mathbb{N}$-graded ring and $T$ is a multiplicative set s.t. $\forall t \in T$ $t$ is homogeneous, then

$$T^{-1}S \text{ is a } \mathbb{Z}\text{-graded ring}$$

where $\quad \deg \frac{f}{g} := \deg f - \deg g$

(check it is well defined as exercise)

$$(T^{-1}S)_0 := \{ x \in T^{-1}S \mid \deg x = 0 \}$$

Given $\mathfrak{p} \in \operatorname{Proj}(S)$, consider $T = \{ f \in S \mid f \text{ homogeneous}, f \notin \mathfrak{p} \}$

We define $\quad S_{(\mathfrak{p})} := (T^{-1}S)_0$

it is the homogeneous version of localizing at $\mathfrak{p}$

it is a local ring with maximal ideal

~~$\mathfrak{p}S_{(\mathfrak{p})} = \{ \frac{s}{t} \} s \in \mathfrak{p}$~~

$$(\mathfrak{p}T^{-1}S)_0 = \{ \tfrac{s}{t} \mid s \in \mathfrak{p},\ s,t \in S_d \text{ for some } d \}$$

**Example:** $S = \mathbb{C}[x,y]$, $\mathfrak{p} = (y)$

$$\underbrace{S_{((y))}}_{\text{important!}} = S_{(\mathfrak{p})} = \{ \tfrac{f}{g} \mid \deg f = \deg g \text{ homog.},\ y \nmid g \}$$

**Check:** $k[x,y]_{((y))} \longrightarrow k\left[y/x\right]_{(y/x)}$

$$\frac{f}{g} \longmapsto \frac{f/x^d}{g/x^d}$$

homogeneous of degree d

**The sheaf:** Given $U \subseteq \mathrm{Proj}\, S$ open

$$T(U) = \{ f \in S \mid f \text{ homogeneous, } f \notin \mathfrak{p} \ \forall \mathfrak{p} \in U \}$$

$\mathcal{O}^{pre}_{\mathrm{Proj}\,S}$ defined as : $U \longmapsto \left(T(U)^{-1} S\right)_0$

$\mathcal{O}_{\mathrm{Proj}\,S}$ is its sheafification

~~The sheaf~~

**Prop.:** $\mathcal{O}_{\mathrm{Proj}(S),\mathfrak{p}} \cong S_{(\mathfrak{p})}$. In particular, $(\mathrm{Proj}\, S, \mathcal{O}_{\mathrm{Proj}\,S})$ is a locally ringed space

Adapt the proof from the Spec case, formally identical

**Corollary:** $\mathcal{O}_{\mathrm{Proj}(S)}$ is given by

$$U \longmapsto \left\{ s : U \longrightarrow \coprod_{\mathfrak{p} \in U} S_{(\mathfrak{p})} \;\middle|\; s \text{ satisfies (1) \& (2)} \right\}$$

(1) for every $\mathfrak{p}$, $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$

(2) for every $\mathfrak{p} \in U$, $\exists$ $V$ open, $\mathfrak{p} \in V \subseteq U$, $a$, $f \in S_d$ for some $d \in \mathbb{N}$ s.t. for every $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{a}{f}$

Proof same as with Spec using the description of the sheafification functor

We want it to be a scheme!

**Remark:** $f \in S_+$ ~~[illegible]~~ homogeneous, i.e. $f \in S_d$ for some $d>0$

$$S \longrightarrow S_f$$

$\uparrow$ $\mathbb{Z}$-grading : $\deg \frac{a}{f^n} = \deg a - \deg f^n$
$= \deg a - nd$

The degree 0 part has a dangerous notation!

$$S_{(f)} := (S_f)_0$$

$\uparrow$ could be confused with ~~[illegible]~~ localizing at $(f)$

**Example:** $S = k[x,y]$ , $f = y$

$$k[x,y]_y = k[x,y,y^{-1}] = k[\tfrac{x}{y}, y, \cancel{y}, y^{-1}] = \left(k[x/y]\right)[y,y^{-1}]$$

So, $S_{(f)} = k[x,y]_{(y)} = \left(k[x,y]_y\right)_0 = k[x/y]$ $\left(\text{Recall } k[x,y]_{((y))} \cong k[y/x]_{\overset{\text{ideal}}{(y/x)}}\right)$

More generally,

$$k[x_0,\dots,x_n]_{(x_i)} = \left(k[x_0,\dots,x_n]_{x_i}\right)_0 \cong k\left[\frac{x_0}{x_i},\dots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\dots,\frac{x_n}{x_i}\right]$$

The idea is that these spectra, which are affine spaces, cover $\mathbb{P}^n$

We will do it in general!

Let $S$ be $\mathbb{N}$-graded, $f \in S_+$ ~~of degree~~

$$D_+(f) := \operatorname{Proj} S \setminus V((f)) \quad \text{open}$$

we have $S_{(f)} = (S_f)_0 \hookrightarrow S_f \leftarrow S$ ~~inducing~~

$$\begin{tikzcd}
\operatorname{Spec} S & \operatorname{Spec} S_f \arrow[l, "\text{open}"'] \arrow[r, "\text{dominant}", "\text{see sheet 4}"'] \arrow[d, phantom, "\cong"] & \operatorname{Spec} S_{(f)} \\
& D(f) & \\
\operatorname{Proj} S \arrow[uu] & D_+(f) \arrow[l] \arrow[u] \arrow[ruu] &
\end{tikzcd}$$

Want: $D_+(f) \cong \operatorname{Spec} S_{(f)}$ as with ring variables

**Issue:** $S$ may not be and so algebra generated in degree 1

**Example:** $S = \Bbbk[x,y,z]$ s.t. $\deg x = 2$, $\deg y = \deg z = 1$

$$S = \Bbbk \oplus \langle y, z \rangle \oplus \langle x, y^2, yz, z^2 \rangle \oplus \langle xy, xz, y^3, \dots, z^3 \rangle \oplus \dots$$

↑

do not generate as ~~a~~ $\Bbbk$-algebra, miss $x$

in this case, $S_{(x)} = (S_x)_0 = \Bbbk\left\langle \frac{y^2}{x}, \frac{yz}{x}, \frac{z^2}{x} \right\rangle \cong \Bbbk[a,b,c]/(ac-b^2)$

Want a way to reduce to deg 1, ~~eower~~

**Def.:** Let $S$ be $\mathbb{N}$-graded, $d \geq 1$, we define the Veronese subring $S^{(d)} := \bigoplus_{n \geq 0} S_{nd}$.

**Proposition:** $\operatorname{Proj} S \cong \operatorname{Proj} S^{(d)}$ as locally ringed spaces

**Sketch:** $\varphi : S^{(d)} \to S$ the inclusion

we have $f : \operatorname{Proj} S \longrightarrow \operatorname{Proj} S^{(d)}$

homog prime $\mathfrak{p}$ $\longmapsto \varphi^{-1}(\mathfrak{p}) = \mathfrak{p} \cap S^{(d)}$

There is a bijection between homogeneous radical ideals

$$I \longmapsto \varphi^{-1}(I) = I \cap S^{(d)}$$

$$\sqrt{J \cdot S} \longleftarrow J$$

so $f$ homeomorphism

if $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$, one can check the natural map

$$S^{(d)}_{\mathfrak{p}} \longrightarrow S_{\mathfrak{q}}$$

induces isomorphism $S^{(d)}_{(\mathfrak{p})} \xrightarrow{\sim} S_{(\mathfrak{q})}$ idea: if $d=2$, poly ring, $\frac{x}{y} = \frac{x^2}{y^2}$

So, it follows that sheaves match

**Example:** $S = \Bbbk[x,y]$, $S^{(2)} = \Bbbk \oplus \langle x^2, xy, y^2 \rangle \oplus \langle x^4, \dots \rangle$

$\cong \Bbbk[z_0, z_1, z_2]/(z_0 z_2 - z_1^2)$ same proj

**Lemma:** If $f \in S_1$, $S_f \cong S_{(f)}[\overset{\deg 1}{x}, x^{-1}]$

Proof: For every $m \in \mathbb{Z}$, we have an isomorphism

$$\begin{tikzcd} (S_f)_m \arrow[r] & (S_f)_0 \\ a \arrow[r, mapsto] & \frac{a}{f^m} \\ f^m b & b \arrow[l, mapsto] \end{tikzcd}$$

So,

$$\begin{tikzcd} S_f \arrow[r, leftrightarrow, "\sim"] & (S_f)_0[x, x^{-1}] \\ a \in (S_f)_m \arrow[r, mapsto, "\cancel{\frac{a}{f^m} x^m}"] & \frac{a}{f^m} x^m \\ b f^m & b x^m \arrow[l, mapsto] \end{tikzcd}$$

Consequence: if $f \in S_1$, via $S \longrightarrow S_f \supset (S_f)_0$

we have 1-1 correspondence between:

~~(1) homogeneous radical prime ideal I in S with $f \notin I$~~

~~(2) homogeneous radical ideals in $S_f$ not containing $f$~~

(1) $I \subsetneq S$ homogeneous, ~~$f \notin \sqrt{I}$~~ radical, $f \notin I$

(2) $J \subsetneq S_f$ homogeneous radical

(3) ~~ideals~~ $\tilde{J} \subsetneq S_{(f)}$ ~~ideal~~ radical $(J = \tilde{J}[x, x^{-1}])$

Thus,
$$\alpha : D_+(f) \longrightarrow \operatorname{Spec}((S_f)_0)$$
$$\mathfrak{p} \longmapsto (\mathfrak{p} S_f)_0 = \left\{ \frac{a}{f^n} \,\middle|\, a \in \mathfrak{p} \cap S_n \right\}$$
is a homeomorphism

**Proposition:** $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ is a scheme and for every $f \in S_+$,

$$(D_+(f), \mathcal{O}_{\operatorname{Proj} S}|_{D_+(f)}) \cong (\operatorname{Spec} S_{(f)}, \mathcal{O}_{\operatorname{Spec}(f)})$$

**Sketch:** Let $f \in S_+$, say $\deg f = d$.
By replacing $S$ by $S^{(d)}$, WMA $\deg f = 1$
By the previous work, we have the homeomorphism

we need $\alpha^\#: \mathcal{O}_{\operatorname{Spec} S_{(f)}} \longrightarrow \alpha_* \mathcal{O}_{\operatorname{Proj}(S)|D_+(f)}$

if $V \subseteq \operatorname{Spec} S_{(f)}$, given $s: V \longrightarrow \coprod_{q \in V} (S_{(f)})_q$

we assign $t: \alpha^{-1}(V) \longrightarrow \coprod_{p \in \alpha^{-1}(V)} S_{(p)}$

in the tautological way induced by

$$(S_{(f)})_q \xrightarrow{\sim} S_{(p)}$$

$$\left(\frac{a}{f^m}\right) \Big/ \left(\frac{b}{f^n}\right) \longmapsto \frac{af^n}{bf^m}$$

for homogeneous $a, b \in S$, $\deg a = m$, $\deg b = n$, $\frac{b}{f^n} \notin q = (pS_f)_0$, i.e. $b \notin p$
since $b \notin p$, $f \notin p$, also $bf^m \notin p$

**Example:** $S = k[x_0, \dots, x_n]$, $I$ homogeneous
$I_0 = I_1 = 0$

↑ otherwise all ↖ otherwise go down one variable

Let $\overline{x_i}$ denote the class of $x_i$ in $S/I$. Then, write

$(I^e)_0 \subseteq k[x_0, \dots, x_n]_{(x_i)}$ for $I \cdot k[x_0, \dots, x_n]_{x_i}$

Let also $I = (f_j)_{j \in J}$, $f_j$ homogeneous of degree $d_j$.

Then, $\left(S/I\right)_{(\bar{x}_i)} \cong k\left[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right] / (I^e)_0$

and $(I^e)_0 = \langle f_j / x_i^{d_j} \rangle_{j \in J}$

This is dehomogeneization

<u>Definition:</u> $A$ a ring

$$\mathbb{P}^n_A := \operatorname{Proj} A[x_0, \dots, x_n]$$

it is covered by $n+1$ copies of $\mathbb{A}^n_A$

<u>Note:</u>
- if $R = R_0$, $\operatorname{Proj} R = \operatorname{Spec} R$
- in general, if $R$ is $\mathbb{N}$-graded, the injection

$$R_0 \longrightarrow R$$

induces $\operatorname{Proj} R \longrightarrow \operatorname{Spec} R_0$

If $V$ is a projective variety with homogeneous coordinate ring $S(V)$, then

$$t(V) = \operatorname{Proj} S(V)$$

## Properties of schemes:

First, we want to define when a scheme looks as nice as possible. For that, we need a lemma from algebra, which is always very useful

**Lemma (Nakayama):** Let $R$ be a ring, $M$ a f.g. module, $I$ an ideal.
If $IM=M$, then $\exists r\in R$ with $\bar{r}=\bar{1}\in R/I$ s.t. $rM=\{0\}$.
(equivalently, $\exists s\in R$ s.t. $sm=m$ for all $m\in M$ $(s=1-r)$)

In particular, if $(R,\mathfrak{m})$ is local and $\mathfrak{m}M=M$, then $M=\{0\}$.

Proof: take $r$ above, ~~then $1-r$ is a unit (otherwise $1-r$ belongs to a maximal ideal, so $1-r\in\mathfrak{m}$, $1=r+(1-r)\in\mathfrak{m}$).~~
then $r$ is a unit as $r\notin\mathfrak{m}$ (indeed, if $r$ not a unit, $(r)\subseteq\mathfrak{m}$ for some max ideal, so $r\in\mathfrak{m}$, $\lightning$), so $M=\{0\}$

In particular, if $\langle\bar{a}_1,\dots,\bar{a}_n\rangle=M/\mathfrak{m}M$, then $M=\langle a_1,\dots,a_n\rangle$
Proof:

$N:=\langle a_1,\dots,a_n\rangle$ over $R$

$N\hookrightarrow M\longrightarrow M/\mathfrak{m}M$, $N\longrightarrow M/\mathfrak{m}M$ surjective, so

$M=N+\mathfrak{m}M$, and

$\mathfrak{m}(M/N)=(\mathfrak{m}M+N)/N=M/N$, so by above $M/N=\{0\}$

Think: implicit function theorem
$R/\mathfrak{m}$ field, lift generators of $M/\mathfrak{m}M$ to generators of $M$
(think of $R$ local ring of a point, so generate $M$ in a nbh of $P$)

The lemma is more general, if $R$ not local need to talk about Jacobson radical

**Def.:** $(R,\mathfrak{m})$ Noetherian local ring, $k = R/\mathfrak{m}$
'$A$ is regular if $\dim A = \dim_k (\mathfrak{m}/\mathfrak{m}^2)$'

**Note:** • makes sense, both finite since $R$ Noetherian local

• always $\dim R \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$, we say

$$\dim R = \min\{n \in \mathbb{N}_{\geq 1} \mid \sqrt{(r_1,\dots,r_n)} = \mathfrak{m}\}$$

take $(\overline{r}_1,\dots,\overline{r}_l)$ a basis of $\mathfrak{m}/\mathfrak{m}^2$ as $R/\mathfrak{m}$ v.s.
Nakayama $\Longrightarrow$ $\langle r_1,\dots,r_l\rangle = \mathfrak{m}$, so $\dim R \leq l$

**Prop.:** $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \#$ minimal set of generators of $\mathfrak{m}$

**Proof:** clearly $\leq$, as a set of gen of $\mathfrak{m}$ descend to gen of $\mathfrak{m}/\mathfrak{m}^2$
$\geq$ above computation by Nakayama

**Interpretation:** in $\mathbb{R}^n$, take $n$ $\mathcal{C}^\infty$ functions $f_1,\dots,f_n$ with $f_i(\underline{0}) = 0$. Look at $\left(\frac{\partial f_i}{\partial x_j}(\underline{0})\right)_{i,j}$

$(f_1,\dots,f_n)$ give a coordinate system in a nbh of $\underline{0}$ $\Longleftrightarrow$ $\left(\frac{\partial f_i}{\partial x_j}(\underline{0})\right)_{i,j}$ invertible $\Longleftrightarrow$ basis of tangent space

inverse fun thm

$\Updownarrow$ dual

differentials $df_i$ basis of cotangent space
$(\cong \mathfrak{m}_{M,x}/\mathfrak{m}_{M,x}^2)$

recall

$$\begin{aligned} T_x M &\xrightarrow{\sim} \left(\mathfrak{m}_{M,x}/\mathfrak{m}_{M,x}^2\right)^\vee \\ D &\longmapsto (f \longmapsto D(f)) \end{aligned}$$

manifold

$\mathfrak{m}/\mathfrak{m}^2$ is the analog of the dual of the tangent space and Nakayama's theorem is the algebraic analog of IFT

<u>Example:</u> • $R = k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$ local ring at origin of $\mathbb{A}^n_k$

$$\mathfrak{m}/\mathfrak{m}^2 \cong (x_1, \dots, x_n)/(x_1, \dots, x_n)^2$$
$$\cong k\langle \bar{x}_1, \dots, \bar{x}_n \rangle$$

• $R = k[u,v,w]/(uw - v^2)$ our round cone

$R_{(u,v,w)}$ local, $\mathfrak{m}/\mathfrak{m}^2 = \dfrac{(u,v,w)}{(uw-v^2)+(u,v,w)^2}$

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 3, \ \dim R_{(u,v,w)} = 2$$

Now we want to use this notion to say when a scheme is nice

<u>Def.:</u> A scheme $X$ is <u>locally Noetherian</u> if it can be covered by $\{\operatorname{Spec} A_i\}_{i \in I}$, each $A_i$ Noetherian. $X$ is Noetherian if the cover may be taken finite.

<u>Rmk.:</u> • in this case, $\mathcal{O}_{X,x} \cong \mathcal{O}_{\operatorname{Spec} A_i, x} \cong (A_i)_\mathfrak{p}$ is Noetherian local

• ~~To need more by [illegible]~~

• we'll see: $X$ locally Noetherian iff for every $\operatorname{Spec} A \subseteq X$ open, $A$ Noetherian

So, we can use our regularity notion for locally Noeth. schemes

<u>Def.:</u> Let $X$ be a locally Noetherian scheme, $x \in X$ with local ring $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ and residue field $k(x)$.

Then, the $k(x)$ vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$ is the Zariski cotangent space at $x$, and its dual

$$T_{X,x} := \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^\vee$$

is the Zariski tangent space at $x$.

$X$ is regular at $x$ if $\mathcal{O}_{X,x}$ is regular, equivalently

$$\dim \mathcal{O}_{X,x} = \dim_{k(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim_{k(x)} T_{X,x}$$

$X$ is regular if it is regular at all of its points.

**Example:** $\mathbb{A}^n_k$ is regular

① check ~~w~~ it is regular at every maximal ideal (not easy if $k \neq \bar{k}$, ok by Nullstellensatz if $k = \bar{k}$)

② hard fact: a Noetherian scheme is regular iff regular at its closed points

How can we do ①?

Recall localization and quotient commute:

**Lemma:** $R$ ring, $\mathfrak{p}$ ~~maximal ideal, the~~ prime ideal, the

$$R/\mathfrak{p} \hookrightarrow \mathrm{Frac}(R/\mathfrak{p}) \cong R_\mathfrak{p} / \mathfrak{p}R_\mathfrak{p}$$

in particular, if $\mathfrak{p} = \mathfrak{m}$ maximal

$$R/\mathfrak{m} \cong R_\mathfrak{m} / \mathfrak{m}R_\mathfrak{m}$$

We have discussed it before

**Corollary:** $R$ ring, $\mathfrak{m}$ maximal ideal

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} \mathfrak{m}R_\mathfrak{m} / \mathfrak{m}^2 R_\mathfrak{m} \quad \text{is an iso}$$

So we do not need to localize to compute the dimension of the tangent space

**Note:** if $R$ f.g. $k$-alg & domain

$$\dim R = \dim R_\mathfrak{m}$$ by additivity & maximality of $\mathfrak{m}$

so, we only need to compare $\dim R$ and $\dim_{R/\mathfrak{m}} (\mathfrak{m}/\mathfrak{m}^2)$.

We want to find a way to detect regularity

**Lemma:** $k$ field, $P \in k^n$ $P = (c_1, \dots, c_n) \leftrightarrow \mathfrak{m} = (x - c_1, \dots, x - c_n)$

$$\Phi : k[x_1, \dots, x_n] \longrightarrow k^n$$
$$f \longmapsto \left(\frac{\partial f}{\partial x_i}(P)\right)_{i=1}^n$$

induces an iso: $\hat{\Phi} : \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} k^n$

**Proof:** WMA $P = (0, \dots, 0)$, take $f, g \in \mathfrak{m} = (x_1, \dots, x_n)$

$$\frac{\partial (fg)}{\partial x_i}(0) = f(0)\frac{\partial g}{\partial x_i}(0) + g(0)\frac{\partial f}{\partial x_i}(0) = 0 \quad \text{so } \Phi(\mathfrak{m}^2) = 0$$

and we get $\hat{\Phi}$. Clearly an iso, as $\hat{\Phi}(\bar{x}_i) = \underline{e}_i$, basis sent to basis

**Rmk.:** assuming we may only check regularity at max ideals this shows $\mathbb{A}^n_k$ is regular if $k = \bar{k}$ (all closed pts come from $k^n$ by Nullstellensatz)

Now, assume $P \in V(I)$, want to check whether ~~[illegible]~~ $V(I)$ (seen as $\operatorname{Spec} R/I$) is regular at $P$

**Lemma:** Consider $I \subseteq \mathfrak{m} \subseteq k[x_1, \dots, x_n]$ with $\mathfrak{m} = (x - c_1, \dots, x - c_n)$ (i.e., a closed $k$ point). Write $\mathfrak{n} := \mathfrak{m}/I$, ideal in $k[x_1, \dots, x_n]/I$.

We have an iso of $k$-v.s.

$$\mathfrak{n}/\mathfrak{n}^2 \xrightarrow{\sim} k^n/\Phi(I)$$

**Proof:** up to translation, $\mathfrak{m} = (x_1, \dots, x_n)$

$$\begin{tikzcd}
\mathfrak{m}/\mathfrak{m}^2 \arrow[rr, "\hat{\Phi}", "\sim"'] \arrow[dd, two heads] & & k^n \arrow[dd, two heads] & & \phi(\mathfrak{m}^2) = 0 \\
 & \mathfrak{m}^2 + I/\mathfrak{m}^2 \arrow[ul, phantom, "\cup"] \arrow[rr, "\sim"] & & \Phi(I) = \hat{\Phi}\left(\mathfrak{m}^2 + I/\mathfrak{m}^2\right) \arrow[ul, phantom, "\cup"] \arrow[ur] & \\
\mathfrak{n}/\mathfrak{n}^2 \arrow[rr, red] & & k^n/\Phi(I) & &
\end{tikzcd}$$

exists: $\mathfrak{n}/\mathfrak{n}^2 = (\mathfrak{m}/I)/(\mathfrak{m}/I)^2 \cong (\mathfrak{m}/I)/(\mathfrak{m}^2 + I/I)$

since $\hat{\Phi}$ is an iso, $\mathfrak{n}/\mathfrak{n}^2 \underset{\text{iso thm}}{\cong} (\mathfrak{m}/\mathfrak{m}^2)\Big/\left(\mathfrak{m}^2+I/\mathfrak{m}^2\right) \underset{\hat{\Phi}}{\cong} k^n/\Phi(I)$

So, we get our criterion

**Theorem** (Jacobian criterion for regularity):

Let $R = k[x_1,\dots,x_n]/I$, $I = (f_1,\dots,f_r)$, P a $k$-rational point of Spec R (i.e., $(x_1-c_1,\dots,x_n-c_n) = \mathfrak{m} \supseteq I$). Then, Spec R is regular at P iff

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial x_j}(P)\right) = n - \dim \operatorname{Spec} R = n - \dim R$$

Proof: $\Phi(I) = $ ~~$\langle f_1 \dots$~~ $\langle \Phi(f_1),\dots,\Phi(f_r)\rangle$, so

~~$\dim_k \mathfrak{m}/\mathfrak{m}^2$~~

$$\dim_k \mathfrak{n}/\mathfrak{n}^2 = \dim_k k^n/\Phi(I) = n - \dim_k \Phi(I)$$
$$= n - \mathrm{rk}\left(\left(\frac{\partial f_i}{\partial x_j}\right)(P)\right)$$

$\downarrow$

$\mathfrak{n} = \mathfrak{m}/I$

Rmk.: • People used the Jacobian criterion to say when an embedded algebraic variety was "smooth".
Zariski proved regularity is an abstract notion

so, we have an abstract notion indep of embedding, but we can check it via embeddings

Example: $k = \bar{k}$

① $X = \operatorname{Spec} k[x,y]/(x^3-y^2)$
$\dim X = 1$, $f = x^3 - y^2$
$\nabla f = \langle 3x^2, -2y\rangle$

cusp

if ~~$(x,y)\neq$~~ $P \neq (0,0)$, $P \in X$ $k$-point, both coordinates are $\neq 0$
$\Rightarrow$ X regular at P (even if char $k$ = 2 or 3)
if $P = (0,0)$, not regular

indeed, $\mathfrak{m}_{X,x} = \sqrt{(x)}$ in $\mathcal{O}_{X,x}$ cannot be generated by 1 element (need both $x$ and $y$, as "$y = x^{3/2}$")

② $k[x]/(x^2)$ dimension 0, not a field, so $\mathfrak{m}$ cannot be generated by 0 elements

indeed $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 = k \cdot x$ of dim = 1

seen in $\mathbb{A}^1_k$, $f = x^2$, $\nabla f = \langle 2x \rangle$, vanishes at 0

~~$1 \neq 0$~~ [illegible]

$$\rightsquigarrow \underset{\substack{\uparrow \\ \mathrm{rk}}}{0} \neq \underset{\substack{\uparrow \\ n}}{1} - \underset{\substack{\uparrow \\ \dim}}{0}$$

Remark: • the Jacobian criterion does not say much about general rings:

$\mathbb{Z}$, $\dim \mathbb{Z} = 1$, $\mathbb{Z}$ is regular (since PID):

p prime: $\dim \mathbb{Z}_{(p)} = 1$

$\dim_{\mathbb{F}_p} p\mathbb{Z}_{(p)}/p^2\mathbb{Z}_{(p)} = \dim_{\mathbb{F}_p} p\mathbb{Z}/p^2\mathbb{Z} = 1$

(0) : $\dim \mathbb{Z}_{(0)} = \dim \mathbb{Q} = 0$

$\dim_{\mathbb{Q}} 0 = 0$

- it can be generalized to work for general rings ("Jacobian criterion, done right" stacks project)
- If we have $V((f_1, \dots, f_r)) \subseteq \mathbb{A}^n_k$ regular and we then consider $V((f_1, \dots, f_r)) \subseteq \mathbb{A}^n_{k'}$, $k' : k$ if $\mathrm{char}(k) = 0$, it is still regular, but if $\mathrm{char}(k) = p > 0$ it may not be the case if $k$ is not perfect (i.e., $k \neq k^p$) → $V(y^2 - x^3 - t)$ in char $2, 3 = p$ over $\mathbb{F}_p(t)$

Before introducing more properties, we will introduce some ways to construct new schemes

Def.: An open subscheme of $(X, \mathcal{O}_X)$ is a ~~topological space~~ scheme $(U, \mathcal{O}_U)$ s.t. $U$ is an open subset of $X$ and $\mathcal{O}_U \cong \mathcal{O}_X|_U$.
An open immersion $(f, f^\#): Y \longrightarrow X$ is a morphism that induces an iso with an open subscheme

Example: $X = \operatorname{Spec} A$, $f \in A$, $U = \operatorname{Spec} A_f$

Def.: A closed immersion $(f, f^\#): Y \longrightarrow X$ is a morphism of schemes s.t. $f$ induces a homeomorphism with $\operatorname{im}(f)$, $\boxed{\operatorname{im}(f) \text{ closed}}$ and $f^\#: \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is surjective.
A closed subscheme of $(X, \mathcal{O}_X)$ is an eq. class of closed immersions, where $(f, f^\#): Y \to X \sim (g, g^\#): W \to X$ if $\exists$ iso $(i, i^\#): Y \to W$ s.t. $(f, f^\#) = (g, g^\#) \circ (i, i^\#)$

Example: $I \subseteq A$, $A \twoheadrightarrow A/I$, so all $A_p \twoheadrightarrow (A/I)_p$ (exact)

so $\operatorname{Spec} A/I$ is a closed subscheme of $\operatorname{Spec} A$

(actually all closed subschemes of $\operatorname{Spec} A$ are like this, but one needs to prove it)

Now, we want to talk products

$S$ base scheme, $X, Y$ $S$-schemes

$$\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

Def.: The fiber product $X \times_S Y$ is a scheme s.t. $\exists$ commutative diagram

$$\begin{tikzcd} X \times_S Y \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

s.t. for all $Z$ and morphisms

$$\begin{tikzcd}
Z \arrow[dr, dashed, "h"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X\times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}$$

$\exists!$ $h$ making the diagram commute

**Thm.:** The fiber product exists and is unique up to iso.

Note: it is clearly commutative

usual nonsense, check yourselves

We are going to see a very important and common style of proof: do it on affines & check it glues

**Lemma 1:** $S = \operatorname{Spec} R$, $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$

$\Longrightarrow X\times_S Y \cong \operatorname{Spec}(A\otimes_R B)$

Proof:

$$\begin{tikzcd}
Z \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \operatorname{Spec} A\otimes_R B \arrow[r, "p"'] \arrow[d, "q"'] & \operatorname{Spec} A \arrow[d] \\
& \operatorname{Spec} B \arrow[r] & \operatorname{Spec} R
\end{tikzcd}$$

$p, q$ exist by

$A \longrightarrow A\otimes_R B$

$a \longmapsto a\otimes 1$

$B \longrightarrow A\otimes_R B$

$b \longmapsto 1\otimes b$

By Ex 2 sheet 3, equiv to

$$\begin{tikzcd}
\Gamma(Z, \mathcal{O}_Z) & & \\
& A\otimes_R B & A \arrow[l] \arrow[ull] \\
& B \arrow[u] \arrow[uul] & R \arrow[l] \arrow[u]
\end{tikzcd}$$

by the commutativity on $R$,

$A \to \Gamma(Z, \mathcal{O}_Z)$

$B \to \Gamma(Z, \mathcal{O}_Z)$

are of $R$-algebras

$\overset{\text{univ prop.}}{\Longrightarrow}$ $\exists!$ $A\otimes_R B \longrightarrow \Gamma(Z, \mathcal{O}_Z)$

again by ex 2 sheet 3, gives unique $h$

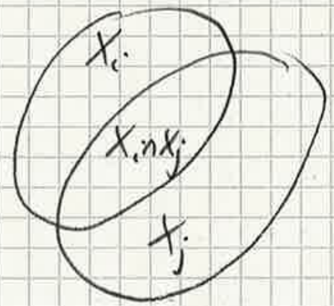
**Lemma 2:** $X, Y$ over $S$. Assume $X \times_S Y$ exists, $p : X \times_S Y \to X$, $U \subseteq X$ open. Then $U \times_S Y$ exists, $U \times_S Y \cong p^{-1}(U)$

Immediate, check $p^{-1}(U)$ satisfies the univ property

**Lemma 3 (glueing):** $X, Y$ over $S$, $\{X_i\}_{i\in I}$ open cover of $X$. If $X_i \times_S Y$ exists for all $i$, then $X \times_S Y$ exists

**Proof:**

$$\begin{tikzcd} X_i \times_S Y \supseteq U_{ij} := p_i^{-1}(X_i \cap X_j) \arrow[d, "p_i"] \\ X_i \end{tikzcd}$$



by Lemma 2, $U_{ij} \cong (X_i \cap X_j) \times_S Y \underset{\substack{\uparrow \\ \text{symmetry}}}{\cong} U_{ji}$

we have iso $\varphi_{ij} : U_{ij} \longrightarrow U_{ji}$

by universality, $\varphi_{ji} = (\varphi_{ij})^{-1}$

In general, to glue spaces we also need a cocycle condition

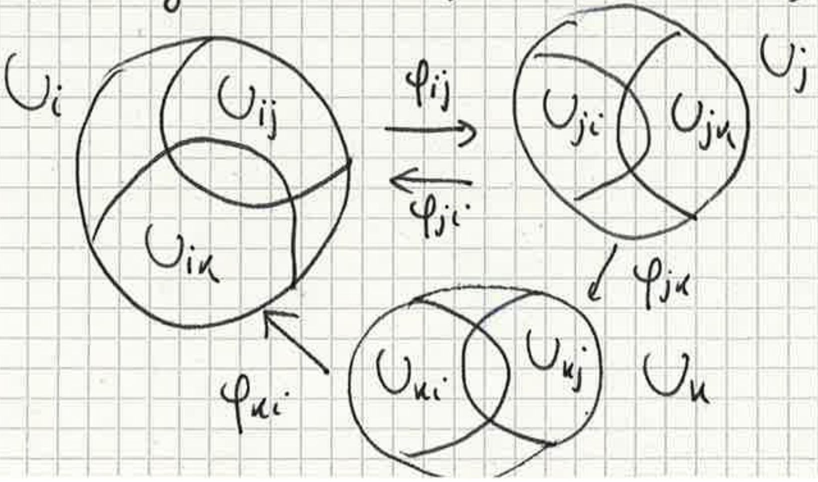
~~On $U_{ij} \cap U_{jk}$ we have, $\varphi_{ij}$~~

~~$U_{ij} \cap U_{ik}$~~

On $U_{ij} \cap U_{ik}$, we have $\varphi_{ij}$ and $\varphi_{ik}$

By Lemma 2, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, where $\varphi_{jk}$ is defined.

Again by Lemma 2, on $U_{ij} \cap U_{ik}$, $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$



we glue them pairwise along the iso but to get sth the gluings need to agree on the overlap

General gluing fact (see Ex II.2.12)
we can glue the $X_i \times_S Y$ to a space $W$

Similarly to sheaves and schemes, since by universality we have $p_j \circ \varphi_{ij} = p_i$, $q_j \circ \varphi_{ij} = q_i$, we get

$$\begin{tikzcd}
W \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d] \\
Y \arrow[r] & S
\end{tikzcd}$$

We need to check $W \cong X \times_S Y$

Given
$$\begin{tikzcd}
Z \arrow[r, "\beta"] \arrow[dr, "\alpha"'] & X \arrow[dr] & \\
& Y \arrow[r] & S
\end{tikzcd}$$
, let $Z_i = \beta^{-1}(X_i)$

$\Rightarrow \exists ! h_i$
$$\begin{tikzcd}
Z_i \arrow[dr, "h_i"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X_i \times_S Y \arrow[r, "p_i"] \arrow[d, "q_i"] & X_i \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}$$

~~By Lemma 2,~~
By gluing, $h_i$ induces $Z_i \xrightarrow{\widehat{h}_i} W$
On $Z_i \cap Z_j$, we have $\tilde{h}_i$ and $\hat{h}_j$. By Lemma 2, they agree
$\Rightarrow$ glue to $h : Z \longrightarrow W$

Check uniqueness locally on the $Z_i$, where $h|_{Z_i} = h_i$

So, $W \cong X \times_S Y$

<u>Proof of</u> <u>theorem:</u> Start with $X$ and $Y$ over $S$

~~By def of scheme, $X$ has open cover by $\{\operatorname{Spec} A_i\}_{i\in I}$~~
~~$Y$ by $\{\operatorname{Spec} B_j\}_{j\in J}$, all R-algs~~

<u>Case 1:</u> $S = \operatorname{Spec} R$

Cover $X = \bigcup_{i\in I} \operatorname{Spec} A_i$, $Y = \bigcup_{j\in J} \operatorname{Spec} B_j$ can by def of scheme (up to iso)

Note: $A_i, B_j$ R-algebras, as inherit morphism to $\operatorname{Spec} R$

$\operatorname{Spec} A_i \times_{\operatorname{Spec} R} \operatorname{Spec} B_j$ exist by Lemma 1

$\Rightarrow X \times_{\operatorname{Spec} R} \operatorname{Spec} B_j$ exist by Lemma 3

$\Rightarrow X \times_{\operatorname{Spec} R} Y$ exist by Lemma 3

<u>Case 2:</u> general $S$, $f: X \to S$, $g: Y \to S$

$S = \bigcup_{\ell\in L} \operatorname{Spec} R_\ell$, $X_\ell = f^{-1}(\operatorname{Spec} R_\ell)$, $Y_\ell = g^{-1}(\operatorname{Spec} R_\ell)$

By Case 1, $X_\ell \times_{\operatorname{Spec} R_\ell} Y_\ell$ exists

Again a gluing argument by uniqueness over

↱ $\operatorname{Spec} R_\ell \cap \operatorname{Spec} R_{\ell'}$ given $X \times_S Y$

Need to show: <u>Lemma 4:</u> $X, Y$ S-schemes, $X \times_S Y$ exists, $V \subseteq$ open in $S$, then

$$f^{-1}(V) \times_V g^{-1}(V) \cong (f \circ p)^{-1}(V) = (g \circ q)^{-1}(V)$$

**Warning:** it is not a product of sets!

e.g.:
$$\begin{tikzcd} \mathbb{A}^n_{\mathbb{R}} \arrow[r] & \operatorname{Spec}\mathbb{R} \\ & \mathbb{A}^m_{\mathbb{R}} \arrow[u] \end{tikzcd}$$

$$\mathbb{A}^n_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{A}^m_{\mathbb{R}} \cong \operatorname{Spec}(\mathbb{R}[x_1,\dots,x_n]\otimes_{\mathbb{R}}\mathbb{R}[y_1,\dots,y_m])$$
$$\cong \operatorname{Spec}(\mathbb{R}[t_1,\dots,t_{n+m}]) = \mathbb{A}^{n+m}_{\mathbb{R}}$$

and $\mathbb{A}^{n+m}_{\mathbb{R}}$ is not product as sets of $\mathbb{A}^n_{\mathbb{R}}$ and $\mathbb{A}^m_{\mathbb{R}}$

e.g.: $\mathbb{A}^2_{k}$ has points corresponding to any irreducible curve, while from a product we retrieve only points corresponding to vertical or horizontal curves

$$\left( \text{“} (0) \times (x-\lambda) \rightsquigarrow (x-\lambda) \text{ in } \mathbb{A}^2_{k} \right)$$

**Note:** ~~the fiber~~ · if we have

$$\begin{tikzcd} W \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

and $W \cong X \times_S Y$ and $p, q$ are the corresponding universal maps, we say it is a Cartesian square and write

- if we have an $S$-scheme $X$, and we have $T \to S$, $X \times_S T$ is naturally a $T$-scheme and we call it base change to $T$.

- if $f: X \to Y$ is a morphism, $y \in Y$, consider $k(y) = \mathcal{O}_{Y,y}/m_y$ with the induced $\operatorname{Spec} k(y) \to Y$.
The fiber of $f$ over $y$ is defined as $X_y := X \times_{\operatorname{Spec} k(y)} Y$
$X_y$ is naturally a $k(y)$-scheme

Why fiber?

<u>Prop.:</u> $\underset{\substack{\uparrow \\ \text{a scheme}}}{X_y} \underset{\text{homeom}}{\simeq} \underset{\substack{\uparrow \\ \text{a set with induced } \tau}}{f^{-1}(y)}$

That's why we like $X_y$, also a scheme

<u>Proof:</u> Take open $\operatorname{Spec} B \hookrightarrow Y$ s.t. $y \in \operatorname{Spec} B$

$y \leftrightarrow p \subseteq B$

So, WMA $Y$ affine (this is no work)

Cover $X = \bigcup_{i \in I} \operatorname{Spec} A_i$, consider $f_i : \operatorname{Spec} A_i \longrightarrow \operatorname{Spec} B$

Clearly, $f^{-1}(y) = \bigcup_{i \in I} f_i^{-1}(y)$

By Lemma 3, $X_y = \bigcup_{i \in I} \operatorname{Spec} A_i \times_{\operatorname{Spec} k(y)} \operatorname{Spec} B$

So, reduce to $X = \operatorname{Spec} A$

$$\begin{tikzcd} f : \operatorname{Spec} A \arrow[r] & \operatorname{Spec} B & \cancel{\varphi:} B \arrow[r, "\varphi"] \arrow[d] & A \\ & \operatorname{Spec} k(y) \arrow[u] & \cancel{B/p}\; B_p/pB_p & \end{tikzcd}$$

$$X_y = \operatorname{Spec}\left(A \otimes_B \cancel{B/}B_p/pB_p\right) \cong \operatorname{Spec}\left( \underset{\substack{\shortparallel \\ A_p}}{(\varphi(B \setminus p))^{-1}A} \Big/ \underset{\substack{\shortparallel \\ pA_p}}{\varphi(p) \cdot (\varphi(B \setminus p))^{-1}A} \right)$$

$$= \{ q \text{ prime of } A_p/pA_p \} = \{ q \text{ prime of } A_p,\ q \supseteq pA_p \}$$

$$= \{ q \text{ prime of } A,\ q \supseteq \varphi(p),\ q \cap \varphi(B \setminus p) = \emptyset \}$$

$$= \{ q \in \operatorname{Spec} A \mid \varphi^{-1}(q) \supseteq p,\ \varphi^{-1}(q) \subseteq p \}$$

$$= \{ q \in \operatorname{Spec} A \mid \varphi^{-1}(q) = p \} = f^{-1}(p)$$

Chase similarly for homeomorphism, this shows 1-1

**Def.:** $f: X \to Y$, if $y \in Y$ is a closed point,

$X_y$ is a closed fiber.

If $Y$ is irreducible (sheet 3 ex 7) with generic point $\eta$, $X_\eta$ is <u>the</u> generic fiber

~~If $\mathcal{O}_{\eta/m_\eta}$ is a field (e.g. $Y = \operatorname{Spec} R$, $R$ domain~~ ) $K$

If $K$ denotes ~~$B$~~ the field $\mathcal{O}_{Y,\eta}/m_{Y,\eta}$

$$\begin{tikzcd} & \operatorname{Spec} K \arrow[r] & Y \end{tikzcd}$$

geometric generic fiber

$$\begin{tikzcd} X_{\bar\eta} \arrow[r] \arrow[d] & X_\eta \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "f"] \\ \operatorname{Spec} \bar K \arrow[r] & \operatorname{Spec} K \arrow[r] & Y \end{tikzcd}$$

the word geometric stresses we are over an alg closed field

**Note:**

$$\begin{tikzcd} & & X \arrow[d, "f"] \\ S'' \arrow[r] & S' \arrow[r] & S \end{tikzcd}$$

, $X \times_{S} S'' \cong (X \times_S S') \times_{S'} S''$

by univ property

So, we accomplished another goal from the first day, now fibers of morphism are objects of our theory!

Examples:

- $k = \bar{k}$, $k[t] = k[x^2] \hookrightarrow k[x]$

$$\mathbb{A}^1_k = \operatorname{Spec} k[x] \longrightarrow \operatorname{Spec} k[t] = \mathbb{A}^1_k$$

$$x \longmapsto t = x^2$$

Generic fiber: $k[t]_{(0)} \cong k(t)$ field, generic pt $\operatorname{Spec} k(t)$

$$k(t) = k(x^2)$$

~~$k(x^2) \otimes k[x^2] \otimes_{k(x^2)}$~~ $k(x^2) \otimes_{k[x^2]} k[x] \cong k(x) = k(\sqrt{t})$

So, the generic fiber is $\operatorname{Spec} k(x)$ as $k(x^2)$-scheme

Closed fibers: $y \longleftrightarrow \mathfrak{m}_y = (t-b)$ for $b \in k$

$$X_y = \mathrm{Spec}\left( k[t]/(t-b) \otimes_{k[t]} k[x] \right)$$

$$\underset{t=x^2}{\overset{\downarrow}{=}} \mathrm{Spec}\left( k[x]/(x^2-b) \right) \underset{k=\bar{k}}{=} \mathrm{Spec}\left( k[x]/(x+\sqrt{b})(x-\sqrt{b}) \right)$$

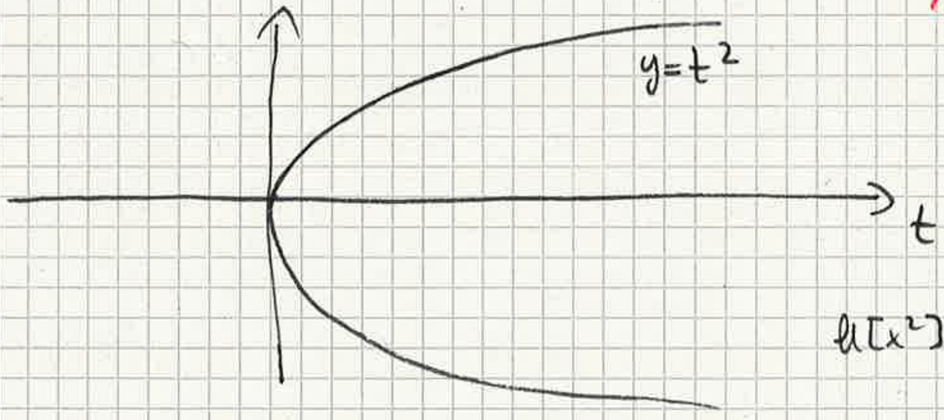
① char $k \neq 2$, so ~~$\sqrt{\ }$~~ $- \neq +$

ⓐ $b \neq 0$ $\quad \mathrm{Spec}\, k[x]/(x+\sqrt{b})(x-\sqrt{b}) = \mathrm{Spec}\left( k[x]/(x-\sqrt{b}) \oplus k[x]/(x+\sqrt{b}) \right)$

$$= \mathrm{Spec}\, k \coprod \mathrm{Spec}\, k$$

2 points

ⓑ $b = 0$ $\quad \mathrm{Spec}\, k[x]/(x^2)$ $\quad$ one point with double multiplicity



$$\begin{tikzcd} k[x^2] \subset k[x] \arrow[d, leftrightarrow] \\ k[t] \subset k[y,t]/(y^2=t) \end{tikzcd}$$

② char $k = 2$, $x^2 + b = (x+\sqrt{b})^2$

all fibers are points with double multiplicity

there pathological / non-geometric things may happen in pos char when ~~the~~ p divides the degree

## What is the geometric generic fiber good for?

$$\overline{k(x^2)} \otimes_{k[x^2]} k[x] \overset{\uparrow}{=} \underbrace{\overline{k(x^2)} \otimes_{k(x^2)} k(x)}$$

associativity of ⊗ (or fiber pr)

do base change from generic fiber

$k(x)$ is a $k(x^2)$ v.s. of rank 2 (degree 2 field ext)

$\Rightarrow \overline{k(x^2)} \otimes_{k(x)} k(x)$ is a $\overline{k(x^2)}$ v.s. of rk 2, but $\overline{k(x^2)}$ alg closed $\Rightarrow$ not a field

~~$\Rightarrow$ Spec $\overline{k(x^2)} \sqcup$ Spec $\overline{k(x^2)}$~~

indeed: $(x\otimes 1)^2 = x^2\otimes 1 = 1\otimes x^2 = (1\otimes x)^2$

$$\Rightarrow \underbrace{(x\otimes 1 + 1\otimes x)}_{\neq 0}\underbrace{(x\otimes 1 - 1\otimes x)}_{\neq 0} = 0$$

char ≠ 2 $\downarrow$ $\Rightarrow \cong \overline{k(x^2)} \oplus \overline{k(x^2)}$ $\Rightarrow$ Spec $\overline{k(x^2)} \sqcup$ Spec $\overline{k(x^2)}$

So, the geometric generic fiber is 2 pts, like ~~all~~ most closed fibers

if char = 2, get $\overline{k(x^2)}[y]/(y^2)$

In general, if $f: X \to Y$ morphism of finite type (we'll define it) over $k = \overline{k}$, $P$ a singularity property (e.g., regular, reduced, ...) (we'll see more)

geometric generic fiber satisfies $P$ $\Longleftrightarrow$ $\exists\, \emptyset \subsetneq U$ open s.t. $\forall y \in U$ closed, $X_y$ satisfies $P$

In this situation, we say the fiber over a general point of $Y$ has $P$

If $k = \overline{k}$: a general point of $Y$ is an ~~closed pt~~ arbitrary closed point $y$ of a specific open $U$ that realizes the property we want to talk about

general $\neq$ generic (all in char 0)

general $\approx$ geometric generic

- $p, q$ primes

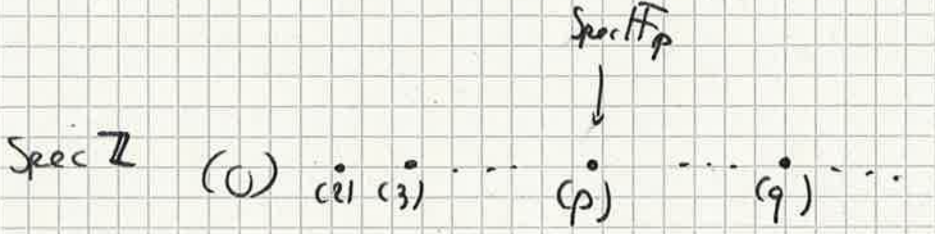
$$\begin{tikzcd} ? \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{F}_p \arrow[d] \\ \operatorname{Spec} \mathbb{F}_q \arrow[r] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

$\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_q =$

① if $p \neq q$, $= 0$, the $0$ ring, $\operatorname{Spec}(\{0\}) = \emptyset$

so, a fiber product can be empty, as a point may have no preimage!

② $p = q$, $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p \cong \mathbb{F}_p \otimes_{\mathbb{F}_p} \mathbb{F}_p = \mathbb{F}_p$



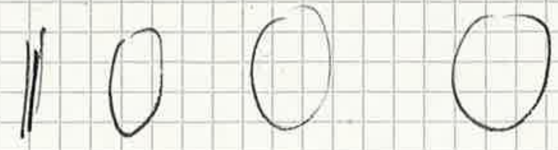
In general, given $f: X \to Y$, can think of $f$ as a "family of fibers"

- $$\begin{tikzcd} \mathbb{P}^2_{\mathbb{Z}} \arrow[d] & \supseteq (x_0^2+x_1^2+x_2^2=0) = X \arrow[dl] \\ \operatorname{Spec} \mathbb{Z} & \end{tikzcd}$$

over $(0)$, $X_{(0)}$ is the conic $(x_0^2+x_1^2+x_2^2=0) \subseteq \mathbb{P}^2_{\mathbb{Q}}$

over $p \neq 2$, $X_{(p)}$ conic in $\mathbb{P}^2_{\mathbb{F}_p}$

over $p=2$, $X_{(2)}$ is the double line $0 = x_0^2+x_1^2+x_2^2 = (x_0+x_1+x_2)^2$



~~the~~ a general fiber is a regular conic

So, we accomplished another promise: regard the same equation over different fields

- $k = \overline{\mathbb{F}_2}$

$$R = \overline{\mathbb{F}_2}[\underset{\substack{\uparrow\\0}}{u},\underset{\substack{\uparrow\\0}}{v},\underset{\substack{\uparrow\\1}}{x},\underset{\substack{\uparrow\\1}}{y},\underset{\substack{\uparrow\\1}}{z}]\Big/(x^2u+y^2v+z^2)$$

$$X = \operatorname{Proj} R = \{x^2u+y^2v+z^2=0\} \subseteq \mathbb{A}^2_k \times_k \mathbb{P}^2_k$$
$$\downarrow$$
$$\operatorname{Spec} R_0 = \operatorname{Spec} \overline{\mathbb{F}_2}[u,v] = \mathbb{A}^2_k$$

Can check $X$ is regular (use Jac criterion on affine patches since $k=\bar{k}$ enough to check $k$ points)

~~as~~ $\eta$ generic point of $\mathbb{A}^2_k$, then $X_\eta$ is regular:
indeed, cover $X$ with affines,

$$\begin{tikzcd} & \operatorname{Spec} A_i \arrow[d] \\ \operatorname{Spec} k(u,v) \arrow[r] & \operatorname{Spec} k[u,v] \end{tikzcd}$$

this is a localization

so ~~since $A_i$ is a domain,~~ $A_i \otimes_{k[u,v]} k(u,v)$ is a localization of $A_i$,
so its local rings are a subsets of local rings of $A_i$

where we check regularity

but $X_{\bar\eta}$ is not!

$$X_{\bar\eta} = \operatorname{Proj} \overline{\mathbb{F}_2(u,v)}[x,y,z]\Big/(ux^2+vy^2+z^2) = \operatorname{Proj} \overline{\mathbb{F}_2(u,v)}[x,y,z]\Big/(x^2t^2+y^2s^2+z^2)$$

take $s,t$ s.t. $s^2=v$, $t^2=u$

$$= \operatorname{Proj} \overline{\mathbb{F}_2(u,v)}[x,y,z]\Big/(xt+ys,z)^2$$

~~not reduced!~~
double line

Similarly, as $\overline{\mathbb{F}_2}$ alg closed, can check closed fibers are double lines!

<u>Def.:</u> Since any scheme maps to Spec $\mathbb{Z}$

(since $\mathbb{Z}$ is initial for rings, ∃! from any affine, by uniqueness they glue)

For any $S$,
$$\mathbb{P}^n_S := \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} S$$
$$\mathbb{A}^n_S := \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} S$$

These extend the ones defined in case $S$ is affine

$$\begin{tikzcd} \mathbb{A}^1_{t,x,y} \arrow[d] & \boxed{x^2-ty^2} \\ \mathbb{A}^1_t & \end{tikzcd}$$

if char 2
reduced
not
geom red

## More properties of schemes

So far: regular, (locally) Noetherian, reduced, irreducible
normal
↳ sheet 6 ↑ sheet 3 ↑ sheet 3

Def.: • $(X, \mathcal{O}_X)$ is connected if $X$ is connected as $\tau$-space
• $(X, \mathcal{O}_X)$ is integral if $\mathcal{O}_X(U)$ is an integral domain for all $U$ open

Irreducibility and connectedness are topological, the other ones are scheme theoretic

Note: if $(X, \mathcal{O}_X)$ is Noetherian as scheme, $X$ is Noetherian as $\tau$-space. Not necessarily vice versa (we saw an example with a Spec)

Theorem: $(X, \mathcal{O}_X)$ is integral if and only if it is irreducible and reduced

Read proof in Hartshorne

~~Theorem: Assume $X$ is connected. Then $(X, \mathcal{O}_X)$ is integral if and only if $x$ is $\mathcal{O}_{X,x}$ is an integral domain for all $x \in X$.~~

Note: if $A, B$ integral domains
$\operatorname{Spec}(A \times B) = \operatorname{Spec} A \amalg \operatorname{Spec} B$
all stalks integral but $\mathcal{O}_{\operatorname{Spec}(A\times B)}(\operatorname{Spec} A \times B) = A \times B$
not integral domain

So, ~~if we assume X is connected~~, being integral cannot be checked on stalks (unless one has extra hypotheses)

**Examples:** • $X = \operatorname{Spec} R$

irreducible $\Leftrightarrow$ $\sqrt{(0)}$ prime   minimal primes corresp to irr comp

reduced $\Leftrightarrow$ $\sqrt{(0)} = 0$   no nilpotents

integral $\Leftrightarrow$ $\sqrt{(0)} = 0$ prime

- $\operatorname{Spec} k[x,y]/(xy)$

reduced, not irreducible

- $\operatorname{Spec} k[x]/(x^2)$   irreducible, not reduced

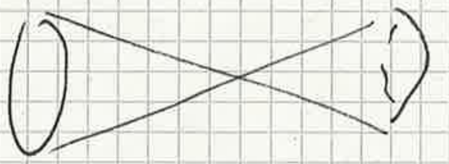
- $\operatorname{Spec} k[x,y,z]/(x^2+y^2+z^2)$

char $k = 2$:  not reduced, but irreducible

$$x^2+y^2+z^2 = (x+y+z)^2$$

$$\left(k[x,y,z]/(x^2+y^2+z^2)\right)/\mathrm{Nil} \cong k[x,y,z]/(x+y+z) \cong k[u,v]$$

char $k \neq 2$



$x^2+y^2+z^2$ is irreducible

$k[x,y,z]$ integral domain

$\Rightarrow$ integral

**Prop.:** $(A,\mathfrak{m})$ regular local ring $\Rightarrow$ $A$ integral domain

**Key lemma:** $(A,\mathfrak{m})$ regular local, $r \in \mathfrak{m} \setminus \mathfrak{m}^2$

$\Rightarrow (A/(r), \mathfrak{m}/(r))$ regular local

check it with def regularity, not bad

the idea is that elt's of $\mathfrak{m} \setminus \mathfrak{m}^2$ can be completed to generators of $\mathfrak{m}$, giving local coordinates, so we are suppressing a direction that is "linearizable" near our point

Fact: if $(X, \mathcal{O}_X)$ is Noetherian, then $X$ is integral iff $X$ is connected and all $\mathcal{O}_{X,x}$ are integral domains

So, previous prop + Jac criterion give us some ways to check integrality

**Theorem:** $X$ is locally Noetherian if and only if every affine open subset is Noetherian

We will skip the proof and only mention basic ingredients

**Lemma:** $R$ ring, $(1)=(f_1,\dots,f_r)$, $R_{f_i}$ all Noetherian $\Rightarrow R$ Noetherian

Consequence: $\operatorname{Spec}R$ Noetherian scheme $\Longleftrightarrow R$ Noetherian

**Lemma:** $\operatorname{Spec}A, \operatorname{Spec}B \subseteq X$ open
$\Rightarrow \operatorname{Spec}A \cap \operatorname{Spec}B$ is the union of all $U \subseteq X$ open that are principal both in $\operatorname{Spec}A$ and $\operatorname{Spec}B$

This is a useful lemma to know

---

**Def.:** $R \subseteq S$ ring extension, $s \in S$ is integral over $R$ if $\exists P \in R[t]$ monic s.t. $P(s)=0$

$R$ is integrally closed in $S$ if, $\forall s \in S$ integral over $R$, $s \in R$

If $R$ is a domain, we say it is integrally closed if so is in $\operatorname{Frac}(R)$

Sheet 6, ex 1 explores this notion

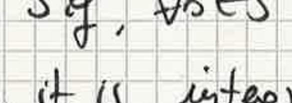
**Def.:** $(X, \mathcal{O}_X)$ is normal if for all $x \in X$, $\mathcal{O}_{X,x}$ is an integrally closed domain

**Proposition:** $R$ UFD $\Rightarrow R$ integrally closed

it is an exercise in definitions

reg loc ring
$\Rightarrow$ UFD
$\Rightarrow$ int closed

Example: $R = k[x,y]/(y^3-x^2)$



$R$ is not integrally closed: $y/x \in \operatorname{Frac}(R) \setminus R$

$P(t) = t^3 - y, \quad P(y/x) = 0$

This suggest this algebraic notion may have something to do with singularities

**Definition:** Let $R$ be a Noetherian ring, $k \in \mathbb{N}$.

(1) $R$ has property $(R_k)$ if for every $\mathfrak{p}$ with $\mathrm{ht}(\mathfrak{p}) \leq k$, $R_\mathfrak{p}$ is regular (i.e., regular in codim $k$)

(2) $R$ has property $(S_k)$ if for every $\mathfrak{p}$ the ring $R_\mathfrak{p}$ has depth at least $\min\{k, \dim R_\mathfrak{p}\}$

↓

in case of local rings $(A, \mathfrak{m})$: maximum length of $(x_1, \dots, x_r) \in \mathfrak{m}$ s.t. $\bar{x}_i$ is not a zero divisor in ~~$A/(x_1,\dots,x_{i-1})$~~

$$A/(x_1, \dots, x_{i-1})$$

**Theorem (Serre):** $R$ Noetherian ring. Then

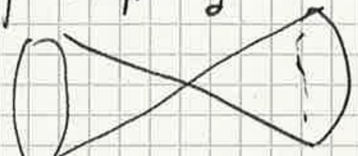
- $R$ reduced $\Longleftrightarrow$ $(R_0) + (S_1)$
- $R$ integrally closed $\Longleftrightarrow$ $(R_1) + (S_2)$

↳ some property of functions

↓ say first instance in ex 6 sheet 6

↓ tells us of the size of the regular locus

**Example:** $R = k[x,y,z]/(z^2 - xy)$, do $k = \bar{k}$ for simplicity



Spec $R$ is $R_1$: Jacobian criterion ~~if $k \neq \bar{k}$~~

check $S_2$ at only bad point

$k[x,y,z]_{(x,y,z)}/(xy - z^2)$ $\quad$ dim $= 2$

take $(x,y)$, length $2$ ✓

Note: $R$ not UFD: $z^2 = z \cdot z = x \cdot y$

**Note:** for a general locally Noetherian scheme $X$, the set of regular points $X_{reg}$ may not be open.
It is open in nice cases: e.g., if $X$ is of finite type over a field (e.g., $\operatorname{Spec} k[x_1,\dots,x_n]/_I$)
Still, even in those cases, it may be empty ($k[x]/(x^2)$)
If $k=\bar{k}$ the Jac criterion proves it for us for $\operatorname{Spec} k[x_1,\dots,x_n]/_I$, as we have to check a rank condition

finite type $\downarrow$ define it next class, don't worry

local noeth ring
reg $\Rightarrow$ UFD $\Rightarrow$ int closed

$\boxed{\text{say reg} \Rightarrow \text{int closed}}$

**Consequence:** $X$ normal of finite type over $k$, $\exists Z \subseteq X$ closed, $\operatorname{codim} Z \geq 2$ s.t. $X \setminus Z$ is regular

So, for curves, Serre's criterion tells us being regular and being normal is the same thing!

**Theorem:** Let $R$ be a Noetherian local ring. TFAE:

① $R$ is an integrally closed domain of dimension $1$

② $R$ is regular of dimension $1$

③ $R$ is a discrete valuation ring (DVR)

↳ explain now!

In this setup, we have $(R, m)$, $m=(t)$ (any such $t$ is called local parameter/uniformizer), $R^* = R \setminus m$, any $r \in R$ is $r = ut^a$, $u \in R^*$, $a \in \mathbb{N}$, any $I = m^k$ $k \in \mathbb{N}$ (so, PID)

**Def.:** Let $K$ be a field. A discrete valuation $v$ on $K$ is $v: K\setminus\{0\} \longrightarrow \mathbb{Z}$ s.t.

(1) $v(ab) = v(a) + v(b)$

(2) $v(a+b) \geq \min\{v(a), v(b)\}$

$R = \{f \in K \mid v(f) \geq 0\} \cup \{0\}$ is the ring associated to $v$, and

$m = \{f \in K \mid v(f) \geq 1\} \cup \{0\}$ is its sole maximal ideal

You can read more in the notes by prof Patakfalvi

**Example:** $k[t]$, t-adic valuation: $v(f) = \mathrm{ord}_t(f)$ (i.e., $f \in (t^{v(f)}), f \notin (t^{v(f)+1})$) } "order of vanishing at 0"

on $k(t)$, $v\left(\frac{f}{g}\right) := v(f) - v(g)$ } "order of 0/pole at 0"

valuation ring: $k[t]_{(t)}$, local ring of $k[t]$ at the origin

In many situations, DVR arise checking "order of vanishing" of functions

---

**Def.:** A morphism of schemes $f: X \to Y$ is locally of finite type if $\exists$ covering $\{\mathrm{Spec}\, B_i\}_{i\in I}$ of $Y$ s.t. for all $i$, $f^{-1}(\mathrm{Spec}\, B_i)$ can be covered by subsets $\mathrm{Spec}\, A_{ij}$, where $B_i \to A_{ij}$ makes $A_{ij}$ a finitely generated $B_i$ algebra.
It is of finite type if, for each $i$, the collection $\mathrm{Spec}\, A_{ij}$ can be taken to be finite.

If $X$ is an $S$-scheme, we say it is (locally) of finite type over $S$ if so is the structure morphism $X \to S$.

**Rmk.:**
- see ex. 3.1, 3.3 in Har, one can go from one cover to any cover
- so, for all $i,j$,

$$\begin{tikzcd} B_i \arrow[r] \arrow[dr] & B_i[x_1, \dots, x_{\ell}] \arrow[d] \\ & A_{ij} \end{tikzcd}$$

- a scheme locally of f.t. $/k$ is covered by $\mathrm{Spec}(k[x_1, \dots, x_r]/I)$

may vary as we move around

So, this notion wants to mean our rings are not too big