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AG II - Schemes and sheaves

Solutions – week 9

Exercise 1. *Nullstellensatz* via *Chevalley.*

(1) Let $\mathfrak{m}$ be maximal in $k[x_1, \ldots, x_n]$. Suppose by contradiction that $\mathfrak{p}_i = k[x_i] \cap \mathfrak{m}$[1] is not maximal, because it is prime, we have $\mathfrak{p}_i = (0)$. The image of the map $\operatorname{Spec}(k(\mathfrak{m})) \to \mathbb{A}^1_{k,x_i}$ is $\mathfrak{p}_i$. But by Chevalley, the image of the map $\operatorname{Spec}(k(\mathfrak{m})) \to \mathbb{A}^1_{k,x_i}$ is constructible, but by our hypothesis, also contains the generic point, and therefore contains an open set. But an open in $\mathbb{A}^1_k$ contains infinitely many points, way much than our singleton $\{\mathfrak{p}_i\}$, leading to a contradiction.

(2) We see by successive quotients, because each $\mathfrak{p}_i$ is maximal in $k[x_i]$, that $(\mathfrak{p}_1, \ldots, \mathfrak{p}_n)$ is maximal. But has it is contained in $\mathfrak{m}$ we have our claimed equality. Also if we denote by $k[x_i]/(\mathfrak{p}_i) = k(\alpha_i)$ where α_i is therefore an algebraic element over k. Then

$$k[x_1, \ldots, x_n]/\mathfrak{m} = k(\alpha_1, \ldots, \alpha_n)$$

and therefore a finite extension.

(3) First, note that from the last point, we deduce that every residue field of a finite type k-algebra at a closed point is a finite extension of k. Let $\mathfrak{m}$ be maximal in $\operatorname{Spec}(B)$. Then we have injections

$$k \to A/f(\mathfrak{m}) \to B/\mathfrak{m}.$$

Because $B/\mathfrak{m}$ is finite dimensional over k, so is $A/f(\mathfrak{m})$. But then the multiplication by every non-zero element is injective, but then surjective because it is a self of a finite dimensional k-vector space. We conclude that $A/f(\mathfrak{m})$ is a field, leading to the desired conclusion.

(4) It suffices to show that every element which is not nilpotent is contained in some maximal ideal. If f is not nilpotent, then $A_f \neq 0$. So there is a maximal ideal in A_f. By the previous point, the preimage of this ideal is maximal in A, concluding.

Exercise 2. *Dual.*

For (2) and (3), the key is to consider the *natural maps* in the sense that for any map $\mathcal{E} \to \mathcal{E}'$ we have commuting diagrams

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{E}^{\vee\vee} \\ \downarrow & & \downarrow \\ \mathcal{E}' & \longrightarrow & \mathcal{E}'^{\vee\vee} \end{array} \qquad \begin{array}{ccc} \mathcal{E}'^{\vee} \otimes_{\mathcal{O}_X} \mathcal{F} & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}', \mathcal{F}) \\ \downarrow & & \downarrow \\ \mathcal{E}^{\vee} \otimes_{\mathcal{O}_X} \mathcal{F} & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \end{array}$$

[1] This is the projection to the i-th coordinate.

To show that these natural horizontal maps are isomorphisms, we can prove that the map is an isomorphism locally. But then, locally these shaves are isomorphic to a finite sum of $\mathcal{O}$, where for those the statement follows from standard linear algebra. Using the above squares, we get the general claim.

Exercise 3. *Compatibilities between f^*, f_* and $\otimes$.*
For (1), one may show first that

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}\otimes, \mathcal{G}, \mathcal{H}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H})).$$

Then the claim follows by combining adjunctions.
For (2), we proceed as in the previous exercise, meaning we construct a natural morphism between the implicit functors in $\mathcal{E}$, and then using this naturality we are allowed to show the claim locally. The natural map correspond by adjunction to tensoring the counit map $f^*\mathcal{E} \otimes (f^* f_* \mathcal{F} \to \mathcal{F})$.

Exercise 4. *Fibre dimension (of coherent sheaves).*
Because each question is local, say $X = \mathrm{Spec}(A)$, where A is Noetherian and we work with M global sections of $\mathcal{F}$, which is a finitely generated A-module. Let $\mathfrak{p} \in \mathrm{Spec}(A)$. For (1), note that if $M(\mathfrak{p})$ is of dimension n, say with basis $m_1, \ldots, m_m$, then we have a surjective map by Nakayama

$$A_{\mathfrak{p}}^n \to M_{\mathfrak{p}}.$$

Find some $a \in A$ such that this surjections lifts to a map

$$A_a^n \to M_a.$$

The coker K of this map is finitely generated and satsfies $K_{\mathfrak{p}} = 0$. Therefore we may localize further to have $K_b = 0$ for some $b \in A$ and concluding that we have a surjective map

$$A_b^n \to M_b.$$

This implies that complements of sets in the statement are open.
For (2), note that φ is continuous to the discrete topology on $\mathbb{N}$ if $\mathcal{F}$ is locally free. Therefore only one fiber can be non-empty because fibers are disjoint opens and the union of all fibers cover the space.
As for (3), proceed as in (1) to get a surjective map

$$A_b^n \to M_b.$$

An element in the kernel is a vector $(a_1, \ldots, a_n)$ where each element is in every prime ideal of A_b. Indeed, for any prime ideal $\mathfrak{p}$ of A_b, looking at

$$k(\mathfrak{p})^n \to M(\mathfrak{p})$$

we have a surjective map between to $k(\mathfrak{p})$-vector spaces of the same dimension so also injective. Because A_b is reduced, the intersection of all primes is the zero ideal, concluding.

Exercise 5. *Fibre dimension (of finite type morphisms)* We recall some results along the way that you can assume.

Lemma 1 (Krull's height theorem)**.** *Let R be a Noetherian ring. Suppose that $\mathfrak{p}$ is a minimal prime of $(f_1, \ldots, f_n)$. Then*

$$\mathrm{ht}(\mathfrak{p}) \leq n.$$

(1) Let R be a Noetherian ring and $\mathfrak{p}$ be a prime ideal. Using Krull's height theorem, show by induction on the height that for every prime $\mathfrak{p}$ of height n there is $(f_1, \ldots, f_n) \subset \mathfrak{p}$ such that $\mathfrak{p}$ is a minimal prime of $(f_1, \ldots, f_n)$ and every minimal prime of $(f_1, \ldots, f_n)$ has height n.

(2) Let $f\colon X \to Y$ be a morphism between locally Noetherian schemes and $Y' \subset Y$ a closed irreducible subset. Show that for every irreducible component $Z \subset f^{-1}(Y')$ that dominates Y' we have

$$\mathrm{codim}(Z, X) \leq \mathrm{codim}(Y', Y).$$

Hint: This is a local problem so you can reduce to affines and use item (1).

Lemma 2. *Let k be a field, A be a finite type k-algebra which is also a domain and $\mathfrak{p} \in \mathrm{Spec}(A)$. Then*

$$\dim(A) = \mathrm{trdeg}_k(\mathrm{Frac}(A))$$
$$\text{and } \mathrm{codim}(\mathrm{Spec}(A/\mathfrak{p}), \mathrm{Spec}(A)) = \mathrm{ht}(\mathfrak{p}) = \dim(A) - \dim(A/\mathfrak{p}).$$

(3) Let $f\colon X \to Y$ be a map between finite type integral k-schemes. Show that for every $y \in f(X)$ and Z irreducible component of X_y we have

$$\dim(X) - \dim(Y) \leq \dim(Z) \leq \dim(X).$$

Hint: Use item (2) with $Y' = \overline{\{y\}}$. Use lemma 2 and the additivity of transcendence degree with $k \mid k(y) \mid K(Z)$. Namely

$$\mathrm{trdeg}_k(K(Z)) = \mathrm{trdeg}_k(k(y)) + \mathrm{trdeg}_{k(y)}(K(Z)).$$

(4) Let $f\colon X \to Y$ be a dominant map between finite type integral k-schemes. Show that there is an open dense $U \subset X$ such that for all $y \in f(U)$ we have $\dim(X_y) = \dim(X) - \dim(Y)$ and $f(U)$ is open.
Hint: show that you can reduce to the affine case $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *with $t_1, \ldots, t_e \in B$, where $e = \dim(X) - \dim(Y)$, such that $t_1, \ldots, t_e$ form a transcendence basis of $K(X)$ over $K(Y)$. Then factor the morphism by* $\mathrm{Spec}(A[t_1, \ldots, t_n])$. *Note that* $X \to \mathrm{Spec}(A[t_1, \ldots, t_e])$ *induces a finite morphism at fraction fields and that* $\mathrm{Spec}(A[t_1, \ldots, t_e]) \to \mathrm{Spec}(A)$ *is isomorphic to* $\mathbb{A}^e_A \to \mathrm{Spec}(A)$ *which is open by exercise 2. Use exercise 1.(2) to conclude.*
Remark. You are free to prove the following weaker version of the statement: show that there is an open dense $U \subset X$ such that for all $y \in f(U)$ we have $\dim(U_y) = \dim(X) - \dim(Y)$ and $f(U)$ is open.

(5) Let $f\colon X \to Y$ be a dominant map between finite type integral k-schemes. For $h \in \mathbb{N}$, let E_h be the set of points x of X such that

there is an irreducible component of $X_{f(x)}$ with dimension at least h, which contains x. Show that E_h is closed.[2]

Hints: If $h \leq e$, then $E_h = X$ by (3). If $h > e$, note that $E_h \subset X \setminus U$ where U is the open of item (4). Proceed by induction on the dimension of X.

(6) Let $f \colon X \to Y$ be a closed map between finite type integral k-schemes. For $h \in \mathbb{N}$, let F_h be the set of points of y of Y such that there is an irreducible component of X_y with dimension at least h. Show that F_h is closed.

Hint: Show that $F_h = f(E_h)$.

This exercise was hand in in a previous year and therefore solutions are attributed to students who wrote them.

(1)(Joel) Suppose we have proved the statement for $n = k$, and let $\mathfrak{p}$ be a prime of height $k+1$. Choose a prime $\mathfrak{q} \subset \mathfrak{p}$ of height k, so by induction there exist $\{f_1, \ldots, f_k\}$ such that $\mathfrak{q}$ is a minimal prime of $I = (f_1, \ldots, f_k)$ and $\mathrm{ht}(\mathfrak{q}) = k$. Let $\{q_i\}$ be the minimal primes of I (so $\mathrm{ht}(q_i) = k$ by induction). As $\mathrm{ht}(\mathfrak{q}_\mathrm{i}) < \mathrm{ht}(\mathfrak{p})$ for all i, $\mathfrak{p} \not\subseteq \mathfrak{q}_i$, so by prime avoidance $\mathfrak{p} \not\subseteq \cup \mathfrak{q}_i$. Hence, there exists some $f_{k+1} \in \mathfrak{p} \setminus \cup \mathfrak{q}_i$. Define $I' = (f_1, \ldots, f_{k+1})$, and let $\mathfrak{p}'$ be a minimal prime of I'. By Krull's height theorem $\mathrm{ht}(\mathfrak{p}') \leq k+1$. As $\mathfrak{q}_j \subsetneq \mathfrak{p}'$ for some $\mathfrak{q}_j$ (as by our choice of f_{k+1}), we have $\mathrm{ht}(\mathfrak{p}') > \mathrm{ht}(\mathfrak{q}_j)$, and hence $\mathrm{ht}(\mathfrak{p}') = k+1$ for any minimal prime $\mathfrak{p}'$ of I'. As $\mathfrak{p}$ contains all the generators of I' and is of height $k+1$ by assumption, $\mathfrak{p}$ is a minimal prime of I', as else we could fit a minimal prime $\mathfrak{q}'$ of height $k+1$ in $I' \subsetneq \mathfrak{q}' \subsetneq \mathfrak{p}$, contradicting $\mathrm{ht}(\mathfrak{p}) = k+1$.

(2)(Joel) As $\mathrm{codim}(Z, X) = \dim \mathcal{O}_{Z,\eta}$ for the generic point η of Z', and similarly for $Y' \subset Y$, the question is local and we can reduce to the case where $f : X = \mathrm{Spec}\ B \to Y = \mathrm{Spec}\ A$ and $\varphi : A \to B$ is the corresponding ring map. Let $\mathfrak{p} \in \mathrm{Spec}\ A$ be such that $V(\mathfrak{p}) = Y'$ and $Z \subset f^{-1}(\mathfrak{p})$ be an irreducible component of $f^{-1}(\mathfrak{p}) = \mathfrak{p}^e$, where $\mathfrak{p}^e$ denotes the extension of $\mathfrak{p}$ by φ. Suppose $\mathfrak{p}$ has height n, so by part (1) there exist $f_1, \ldots, f_n \in A$ such that $\mathfrak{p}$ is a minimal prime of $I = (f_1, \ldots, f_n)$. Then $\mathfrak{p}^e \supseteq I^e = (\varphi(f_1), \ldots, \varphi(f_n))$. Let $\mathfrak{q}$ be a minimal prime of $\mathfrak{p}^e$ corresponding to the irreducible component Z. Next, we show that $\mathfrak{q} \supseteq \mathfrak{p}^e$ is a minimal prime of I^e, which by Krull's height theorem implies that $\mathrm{ht}(\mathfrak{q}) \leq n = \mathrm{ht}(\mathfrak{p})$, which is equivalent to the inequality $\mathrm{codim}(Z, X) \leq \mathrm{codim}(Y', Y)$.
First, we may assume that $\mathfrak{p}^e \supsetneq I^e$, as if $\mathfrak{p}^e = I^e$, the result is immediate as $\mathfrak{q}$ is a minimal prime of $\mathfrak{p}^e$. Now, suppose there exists $\mathfrak{r} \in \mathrm{Spec}\ B$ such that $\mathfrak{q} \supsetneq \mathfrak{r} \supseteq I^e$. Then $\varphi^{-1}(\mathfrak{q}) \supseteq \varphi^{-1}(\mathfrak{r}) \supseteq \varphi^{-1}(I^e) \supseteq I$. As Z dominates Y', $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$, and furthermore as $\mathfrak{p}$ is a minimal prime of I. As $\mathfrak{q}$ is a minimal prime of $\mathfrak{p}^e = \varphi^{-1}(\mathfrak{r})^e \subseteq \mathfrak{r}$, and $\mathfrak{r}$ is by assumption prime, we see that $\mathfrak{q} = \mathfrak{r}$, and hence $\mathrm{codim}(Z, X) \leq \mathrm{codim}(Y', Y)$.

(3)(Joel) Again, the question is local, so let $f : X = \mathrm{Spec}\ B \to \mathrm{Spec}\ A = Y$ be a morphism of affine schemes, and $y \in Y$ be a point corresponding to

[2]The statement is true for any $f \colon X \to Y$ between X and Y finite type k-schemes without the dominant hypothesis. This can be shown by an easy reduction to the case of the exercise.

$\mathfrak{p} \in \operatorname{Spec} A$. Set $Y' := \bar{\{y\}}$, the closure of y in Y. Let $\phi : A \to B$ be the ring map corresponding to f.
B is a finitely generated $A-$algebra, so $B \bigotimes_A k(y)$ is a finitely generated $k(y)$-algebra, and hence $\dim Z = \operatorname{trdeg}_{k(y)}(K(Z))$ for any any irreducible component Z of $f^{-1}(y)$, as Z is of finite type over $k(y)$ and the trace formula holds.
If Z is an irreducible component of X_y, we want to show that $\bar{Z}$ is an irreducible component of $f^{-1}(Y')$. Z is contained in some irreducible component W of $f^{-1}(Y')$. As $Z \subset W$ and $Z \subset X_y$, the image of W contains $y \in Y$. Let η be the generic point of W. Then $f(W) \subset Y'$ is a dense inclusion with both W and Y' irreducible, and so $f(\eta) = y$, the generic point of Y'. Note that the closure of η intersected with X_y is irreducible and contains Z, and hence $W = \bar{\{\eta\}} \subset \bar{Z}$, so $\eta \in Z$. By part (2) we have $\operatorname{codim}(\bar{Z}, X) \leq \operatorname{codim}(Y', Y)$. As Z is dense in $\bar{Z}$, $K(Z) = K(\bar{Z}$, and similarly for $K(y)$ and $K(Y')$. Using the trace formula for $k - k(y) - k(Z)$, we get $\operatorname{trdeg}_{K(y)} K(Z) = \operatorname{trdeg}_k K(Z) - \operatorname{trdeg}_k K(y) = \operatorname{trdeg}_k K(\bar{Z}) - \operatorname{trdeg}_k K(Y')$. Using the inequality for codimensions we get $\dim X - \dim Y \leq \dim \bar{Z} - \dim Y' = \operatorname{trdeg}_k K(\bar{Z}) - \operatorname{trdeg}_k K(Y') = \operatorname{trdeg}_{K(Y')} K(\bar{Z}) = \operatorname{trdeg}_{K(y)} K(Z) = \dim Z$. As $X_y \simeq f^{-1}(y) \subset X$, we get that $\dim Z \leq \dim X$, and hence in total we have

$$\dim X - \dim Y \leq \dim Z \leq \dim X.$$

(4)(Héloïse) We prove the following.

Lemma. *Let $f : X \to Y$ be a dominant map between finite type integral k-schemes. There is an open dense subset $V \subset X$ such that for all $y \in f(V)$,*

$$\dim(X_y) = \dim(X) - \dim(Y)$$

and $f(V)$ is open.

We begin by proving the following weaker statement.

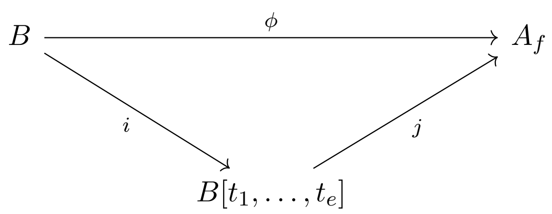
Lemma. *Let $f : X \to Y$ be a dominant map between finite type integral k-schemes. There is an open dense set $U \subset X$ such that for all $y \in f(U)$*

$$\dim(U_y) = \dim(X) - \dim(Y).$$

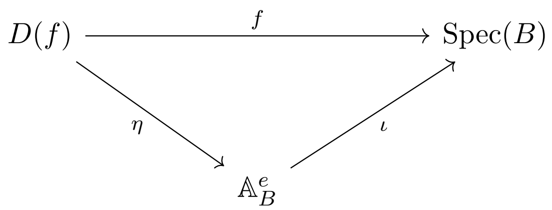
Proof. Note that we are free to reduce Y to an affine dense open and also U can be taken to be inside a dense affine open of X, so we can reduce to the affine case, as we do in what follows.
Proof of the affine case. We denote by $\phi : A \to B$ the ring map corresponding to f. Note that since f is a morphism between k-schemes, ϕ is injective. Let $e := \dim(X) - \dim(Y)$.
By using the additivity of the trenscendance degree (since f is a dominant map between finite type integral k-schemes) to the field extensions $K(X) \mid K(Y) \mid k$ induced by ϕ, we get that

$$\begin{aligned} e &= \dim(X) - \dim(Y) \\ &= \operatorname{trdeg}_k(K(X)) - \operatorname{trdeg}_k(K(Y)) \\ &= \operatorname{trdeg}_{K(Y)}(K(X)). \end{aligned}$$

Let $\{t_1, \ldots, t_e\}$ be a transcendence basis of $K(X)$ over $K(Y)$. Note that the elements $t_i \in K(X)$ may be seen as fractions with numerators in A and denominators in B therefore, by considering f to be the product of all the denominators of the t_i's (which is finite since $e < \infty$), and localizing A at f, we get that the t_i's are elements of A_f. From there, we get the following commutative diagram



which induces the following diagram on affine schemes.



Since we are looking for a dense open subset, we sloppily rename $\mathrm{Spec}(A) := D(f)$ for the rest of the proof. By exercise 2 of sheet 8, the map $\iota : \mathbb{A}^e_B \cong \mathrm{Spec}(B[t_1, \ldots, t_e]) \to \mathrm{Spec}(B)$ is open, while the map $\eta : \mathrm{Spec}(A) \to \mathbb{A}^e_B$ induced by j is dominant since j is injective. Morevoer, η is finite type since it is a map of finite type integral k-schemes. By noting that the field extension

$$K(Y)(t_1 \ldots t_n) \subseteq K(X) = \mathrm{Frac}(A)$$

is finite, since $\mathrm{trdeg}_{K(Y)}(K(X)) = e \leq \infty$, we conclude by exercise 1.2 of sheet 8 that there exists a non-empty open set $V \subset \mathbb{A}^e_B$ such that the restriction of the morphism η to $\eta^{-1}(W)$ is finite and dominant since η is dominant. In particular, it is closed. Then, since the map is dominant and closed, it is surjective.

Moreover, since η is surjective, $\eta(\eta^{-1}(W)) = W$ and therefore, $f(\eta^{-1}(W)) = \iota \circ \eta(\eta^{-1}(W)) = \iota(W)$ which is open since ι is an open map. Finally, since A is an integral domain, it is in particular irreducible and we conclude that $\eta^{-1}(W)$ is dense.

Now for any $\mathfrak{p} \in f(\eta^{-1}(W)) = \iota(W)$, since finitness and surjectivity of a morphism is stable under base change, the morphism g coming from the following base change is finite and surjective.

$$
\begin{array}{ccc}
(\eta^{-1}(W))_{\mathfrak{p}} & \longrightarrow & \eta^{-1}(W) \\
\downarrow g & & \downarrow \eta \\
W \times_Y k(\mathfrak{p}) & \longrightarrow & W \quad le \\
\downarrow & & \downarrow \\
k(\mathfrak{p}) & \longrightarrow & Y
\end{array} \tag{1}
$$

Therefore,

$$\dim(\eta^{-1}(W))_{\mathfrak{p}}) = \dim(W \times_Y k(\mathfrak{p})) \leq \dim(\mathbb{A}^e_{k(\mathfrak{p})}) = e.$$

Now note that $B[t_1, \ldots, t_n]$ is an integral domain, hence $\mathbb{A}^e_B$ is an integral scheme. Therefore, $\dim(W) = \dim(\mathbb{A}^e_B) = \dim(B) + e$. Furthermore, since the restriction of the morphism η to $\eta^{-1}(W)$ is dominant, hence finite and surjective, $\dim(\eta^{-1}(W)) = \dim(W)$.
By applying the result from question 3 to the restriction of f to $\eta^{-1}(W)$, then for any irreducible component Z of the fibre $(\eta^{-1}(W))_{\mathfrak{p}}$, we get

$$\dim(B) + e - \dim(B) \leq \dim(Z).$$

Since the above holds for any irreducible component, we conclude that $e \leq \dim((\eta^{-1}(W))_{\mathfrak{p}})$. Thus

$$\dim((\eta^{-1}(W))_{\mathfrak{p}}) = e$$

and we can pick $U = \eta^{-1}(W)$. □

This lemma does not yet allow to generalize to the the whole fiber X_y, as the equality $\dim(U_y) = \dim(X_y)$ might not hold for any open dense set U. We therefore need to further refine U using the following lemma.

Lemma. *Let $f : X \to Y$ be a map between finite type integral k-schemes. Then, there exists a dense open set $V \subseteq Y$ such that for all $y \in Y$, $U_y \subset X_y$ is dense.*

Proof. Reduction to the affine case. Up to shrinking Y, we may assume that Y is an affine scheme $Y := \mathrm{Spec}(A)$.
Now, suppose that we have proven the statement when X is an affine scheme. For a general scheme X, consider an affine open cover $X = \bigcup_i W_i$ where W_i are affine schemes. For each W_i, there exists an open dense set $V_i \subseteq Y$ such that for any $y \in V_i$, $(U \cap W_i)_y \subset (W_i)_y$ is dense. Consier $V := \bigcup_i V_i$. Then for any $y \in V$, $U_y \subset X_y$ is dense. Indeed, the fiber X_y is a glueing of the $(W_i)_y$'s while U_y is a glueing of the $(U \cap W_i)$'s, where each $(U \cap W_i)_y$ is dense in $(W_i)_y$.

Proof of the affine case. Suppose that $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a map between finite type integral k-schemes. Since the principal open sets form a basis for the topology on $\mathrm{Spec}(B)$, up to shrinking U, we may assume that

U is of the form $D(t)$ with $t \in B$.
Consider the short exact sequence

$$(2) \qquad 0 \longrightarrow B \xrightarrow{\cdot t} B \longrightarrow B/(t) \longrightarrow 0 \ .$$

By the *Generic flatness theorem*, there exists a dense open set $V \subset \operatorname{Spec}(A)$ such that for any $y \in V$, the morphism $B \otimes_A k(y) \to B_t \otimes_A k(y)$ is injective because $B/(t)$ can be supposed to be flat on this open, which implies that the morphism $(D(t))_y \to X_y$ is dominant. □

Let V be as in the previous lemma. By considering $U' := U \cap f^{-1}(V)$, we have proven the general case.
(5) (Alissa) Let $f : X \to Y$ a dominant map between finite type integral k-schemes. For h a positive or 0 integer we define

$$E_{X,h} := \{x \in X | \exists Z \subseteq X_{f(x)} \text{ an irreducible component which contains } x \text{ s.t. } \dim Z \geqslant h\}$$

Show that $E_{X,h}$ is closed.

We see that if $h \leqslant \dim X - \dim Y$ then by part 3 we conclude that $E_{X,h} = X$, hence it is closed. Now if we consider the case $h > \dim X - \dim Y$ then we see that if take the U obtained in Part 4 then $E_{X,h} \subseteq X \setminus U$. We proceed by induction on the dimension of X to prove that $E_{X,h}$ is closed. (take the version of Part 4 with X_y and not only U_y)
Suppose that $\dim X = 0$. Then we see that $0 \leqslant \dim Z \leqslant \dim X_y = \dim f^{-1}(y) \leqslant \dim X = 0$. So $E_{X,h}$ is just $\emptyset$. For the induction step, suppose that the result is true for every X of dimension $d-1$ or less. Suppose that $\dim X = d$. Then if we consider $E_{X,h}$ we see that $E_{X,h} \subseteq X \setminus U$ which is closed. So now we can consider the decomposition of $X \setminus U$ in a union of irreducible closed subsets C_i. The latter will have dimension strictly smaller than X since they are irreducible in X which is itself irreducible. Since X is a fintie type k-scheme, we see that there must be only finitely many C_i's. We would like to show that $E_{X,h} = \bigcup_{i=1}^n E_{C_i,h}$. To do so, notice first that we can endow each C_i with a reduced scheme structure. Since it is irreducible, we get that C_i is integral. If we show the above equality, we would like to use induction since the C_i's have strictly lower dimension than d. However, to apply induction we have to be in the good conditions. So we need an integral image and a dominant map. Furthermore, we need C_i and the image to be finite type k-schemes. So let us consider the morphism $f|_{C_i} : C_i \to \overline{f(C_i)}$ where we endow $\overline{f(C_i)}$ with the reduced scheme structure. Since C_i is irreducible, then $f(C_i)$ is too and so is $\overline{f(C_i)}$. It is direct that the morphism is dominant. Since X and Y are finite type k-schemes, then C_i and $\overline{f(C_i)}$ are finite type k-schemes. As before, $f|_{C_i}$ is finite type since it is a morphism between finite type k-schemes.
First, let $x \in E_{C_i,h}$. We would like to show that $\bigcup_{i=1}^n E_{C_i,h} \subseteq E_{X,h}$. Remember that we have

$$C_{i,f(x)} \approx f^{-1}(f(x)) \cap C_i \subseteq f^{-1}(f(x)) \approx X_{f(x)}$$

Since the C_i's are closed, $f^{-1}(f(x)) \cap C_i$ is a closed subset of $f^{-1}(f(x))$. Hence an irreducible component of $C_{i,f(x)}$ containing x is also an irreducible closed subset of $f^{-1}(f(x))$ containing x.

Now for the other inclusion let $x \in E_{X,h}$. Then we notice that

$$X_{f(x)} \approx f^{-1}(f(x)) = \cup_{i=1}^{n} f^{-1}(f(x)) \cap C_i$$

We see that the irreducible components of $X_{f(x)}$ are the irreducible components of each $f^{-1}(f(x)) \cap C_i$. This is how we get $E_{X,h} \subseteq \bigcup_{i=1}^{n} E_{C_i,h}$.

□

(6) (Alissa) Let $f : X \to Y$ a closed map between finite type integral k-schemes. For $h \in \mathbb{N}$ we define

$$F_h := \{y \in Y | \exists Z \subseteq X_y \text{ an irreducible component s.t. } \dim Z \geqslant h\}$$

Show that F_h is closed.

To show this, we will rather prove that $f(E_h) = F_h$. Since f is closed and using Part 5, it follows immediately that F_h is closed.
We will show that $f(E_h) = F_h$ by showing each inclusion.

$\underline{f(E_h) \subseteq F_h}$: Let $y \in f(E_h)$. Then there exists $x \in E_h$ such that $f(x) = y$. This implies that there exists an irreducible component of $X_{f(x)} = X_y$ of dimension at least h. Hence $f(x) \in F_h$ by definition.

$\underline{F_h \subseteq f(E_h)}$: Let $y \in F_h$. If $y \notin f(X)$ we see that the fiber must be the empty set since $X_y \approx f^{-1}(y)$. Hence we see that $y \in f(X)$. We know that there exists an irreducible component Z of X_y such that it has dimension at least h. Now we only have to prove that Z contains at least one $x \in f^{-1}(y)$. However, we remember that $X_y \approx f^{-1}(y)$, hence necessarily Z contains an element x of $f^{-1}(y)$ and so Z is an irreducible component of $X_{f(x)} = X_y$ of dimension at least h.