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AG II - Schemes and sheaves

Solutions – week 7

Exercise 1. *Finite covers of* $\mathbb{P}^1_{\mathbb{C}}$

(1) The preimage of $D_+(x)$ and $D_+(y)$ is $D_+(x^n) = D_+(x)$ and $D_+(y^n) = D_+(y)$. The map on ring of functions is given the $\mathbb{C}$-algebra map given by sending $\frac{y}{x} \mapsto \left(\frac{y}{x}\right)^n$ and $\frac{x}{y} \mapsto \left(\frac{x}{y}\right)^n$. Now note that $\mathbb{C}[t^n] \subset \mathbb{C}[t]$ is finite free of degree n. A basis being $1, t, \ldots, t^{n-1}$.

(2) The generic fiber is $\mathbb{C}(\frac{x}{y})$ because the generic point's unique preimage is the generic point for dimension reasons. But let us also deduce as follows. Locally this amounts to compute the tensor product=pushout of

$$\begin{tikzcd} \mathbb{C}[x/y] \arrow[r, "x/y \mapsto (x/y)^n"] \arrow[d] & \mathbb{C}[x/y] \\ \mathbb{C}(x/y) & \end{tikzcd}$$

But note that localizing $\mathbb{C}[x/y]$ at the multiplicative $(\mathbb{C}[x/y])^n \setminus 0$ (where the power n here means elements of this ring that are the n-th power) is the same as localizing by the multiplicative subset $\mathbb{C}[x, y] \setminus 0$. Indeed inverting an element or it's n-th power is the same. As for the fiber of closed points, say $(t - \lambda)$ where $t = x/y$ or y/x seen in $D_+(x)$ or $D_+(y)$, we get

$$\operatorname{Spec}(\mathbb{C}[t]/(t^n - \lambda)).$$

So if $\lambda = 0$, we get a single non-reduced point and n-copies of $\operatorname{Spec}(\mathbb{C})$ otherwise.

Exercise 2. (1) Note that as the algebra is finitely generated in *degree 1*, we have that for $n \geq 1$

$$(S_+)^n = \bigoplus_{k \geq n} S_k.$$

The result follows.

(2) Just note that S_0-generators in degree 1 are sent to S-generators in degree 1.

(3) Note that $S_+^n / S_+^{n+1} = S_n$. Therefore, the result follows.

(4) Let $a \in S_1$ be a degree 1 element. We denote by B the blow-up algebra we are working with, and if $f_i \in S_i$ then we denote by $f_{i,(k)} \in B_k$ for this element placed in degree $i \geq k$.

We first remark that $S_{(a)}$ embeds as a ring in $B_{(a_{(1)})}$ by

$$\frac{f_k}{a^k} \mapsto \frac{f_{k,(k)}}{a_{(1)}^k}.$$

Note also that any element in $B_{(a_{(1)})}$ is a sum of elements of the form

$$\frac{f_{i,(k)}}{a_{(1)}^k}$$

where $f_i \in S_i$ with $i \geq k \geq 0$. Now note that in B, if $d = i - k$, we have

$$f_{i,(k)} a_{(1)}^d = f_{i,(i)} a_{(0)}^d = (f_i a^d)_{(d+k)}.$$

In consequence we can express the above as

$$\frac{f_{i,(k)}}{a_{(1)}^k} = \left(\frac{f_{i,(k)}}{a_{(1)}^i} \right) a_{(1)}^d = \left(\frac{f_{i,(i)}}{a_{(1)}^i} \right) a_{(0)}^d.$$

In consequence the $S_{(a)}$-algebra map $S_{(a)}[t] \to B_{(a_{(1)})}$ sending to $t \mapsto a_{(0)}$ is surjective. We claim that it is also injective. Indeed, suppose that (here each i in the sum depends on d but we omit it to simplify the notation)

$$0 = \sum_{d=0}^{n} \left(\frac{f_{i,(i)}}{a_{(1)}^i} \right) a_{(0)}^d$$

Then, there is a $N \geq 0$ such that in $(B)_N = \bigoplus_{k \geq N} S_k$ we have

$$0 = \sum_{d=0}^{n} \left(f_{i,(i)} a_{(1)}^{N-i} \right) a_{(0)}^d = \sum_{d=0}^{n} \left(f_i a^{N-i+d} \right)_N$$

But each individual term of this sum is of degree (seen as elements of S) $d + N$. Therefore, the claim follows.

Remark. Note that we proved more precisely that $B_{(a_{(1)})}$ identifies with

$$\bigoplus_{d \geq 0} \widetilde{S}(d)(D_+(a)).$$

We see therefore that the blow-up of the cone at vertex identifies with $\mathbb{V}(\mathcal{O}_{\mathrm{Proj}(S)}(-1))$, the *canonical bundle on* $\mathrm{Proj}(S)$. You may be able to understand these notations at the end of the lecture.

Exercise 3. (2) Denote by A the integral domain we are now working with, let $f \in K = \mathrm{Frac}(A)$ and denote by I the ideal of denominators. Let $\mathfrak{p} \in V(I)$ a minimal prime. Therefore $\sqrt{IA_{\mathfrak{p}}} = \mathfrak{p}A_{\mathfrak{p}}$. By contradiction, suppose that it is of height at least 2. By the (S2) hypothesis, we see that therefore is a regular sequence $g, h \in \mathfrak{p}A_{\mathfrak{p}}$, and therefore without loss of generality in $IA_{\mathfrak{p}}$. Let also $a, b \in A$ such that

$$f = \frac{a}{g} = \frac{b}{h}.$$

This implies $ha = bh$. As (g, h) is regular, this implies that $a \in (g)$ reducing $\mod (g)$. But then $f \in A_{\mathfrak{p}}$, implying that $1 \in \mathfrak{p}A_{\mathfrak{p}}$, a contradiction.

(3) If $f \in \mathcal{O}_X(U)$ then $U \subset X \setminus V(I)$. So $V(I) \subset X \setminus U$. By the previous point, $V(I)$ is empty.

Exercise 5.

(1) We show that a scheme X is separated if and only if for every (for a cover by) affine opens U and V (or for all pairs of any affine cover) of X
- $U \cap V$ is again affine,
- the natural map $\mathcal{O}_X(U) \otimes \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is surjective.

If X is separated note that $U \cap V$ can be realized as the *closed* subscheme of the affine scheme $U \times V = \mathrm{Spec}(\mathcal{O}(U)) \times \mathrm{Spec}(\mathcal{O}(V))$ which is $(U \times V) \cap \Delta$, where $\Delta \subset X \times X$ is the diagonal and is a closed subscheme by very assumption. This explains that the conditions are necessary. To see that they are sufficient, note that to check that $\Delta \subset X \times X$ is a closed immersion we can check it locally on *any* affine cover $(U_i \times U_j)_{ij}$ of $X \times X$ where $(U_i)_i$ is any affine cover of X. But now we see that the first condition ensures that the intersection $\Delta \cap U_i \times U_j$ is affine, and the second condition tells that it is a closed immersion.

(2) Now for a scheme of the form $\mathrm{Proj}(B)$, it suffices to show it for standard opens of the form $D_+(f)$ and $D_+(g)$ for f and g homogeneous. The intersection is affine, being $D_+(gf)$. We need to show that

$$B_{(f)} \otimes B_{(g)} \to B_{(fg)}$$

is surjective. But any degree zero element $\frac{h}{f^i g^j}$ can be written as a product of degree zero elements $\frac{bh}{f^N} \frac{f^N}{g^M}$ for appropriate $b \in B$ and $N, M \in \mathbb{N}$.

(3) We can use the criterion displayed above. Denote by $\mathbb{A}^1_a$ and $\mathbb{A}^1_b$ the two different copies of $\mathbb{A}^1$ inside this scheme. Their intersection is their common $\mathbb{G}_m$ by construction. It is the second property in the criterion above that fails. Namely

$$k[t] \otimes k[t] \to k[t, t^{-1}]$$

is not surjective, missing t^{-1}. This shows that $X \to \mathrm{Spec}(\mathbb{Z})$ is not separated. As a byproduct $X \to \mathrm{Spec}(k)$ is also not separated. Indeed if it was, as $\mathrm{Spec}(k) \to \mathrm{Spec}(\mathbb{Z})$ is, $X \to \mathrm{Spec}(\mathbb{Z})$ would also.

Exercise 6.

This exercise was to hand in in a previous years so solutions are credited to past students who wrote them.

(1)(Daniil) Let's use the equivalence of categories of rings and affine schemes. So to each diagram from the definition of the action of $\mathbb{G}_m$ on an affine scheme X corresponds one to one with a map of rings $\mu^* \colon A \to A[t, t^{-1}]$ such that

$$\begin{tikzcd}
A[x,x^{-1},y,y^{-1}] & A[t,t^{-1}] \arrow[l, "\mathrm{id}\otimes\mu^*"'] \\
A[t,t^{-1}] \arrow[u, "(t\to xy)\otimes\mathrm{id}"] & A \arrow[l, "\mu^*"] \arrow[u, "\mu^*"']
\end{tikzcd}$$

and the composition of μ with evaluation at 1 (denoted by c) is id.
Write $\mu(a) = \sum_i a_i t^i$. The commutativity of the diagram means

$$\sum_i a_i (yx)^i = ((t \to xy) \otimes \mathrm{id})\,(\sum_i a_i t^i) = ((t \to xy) \otimes \mathrm{id})\,(\mu^*(a))$$

$$\overset{\text{commutativity of the diagram}}{=} (\mathrm{id}\otimes\mu^*)\,(\mu^*(a)) = (\mathrm{id}\otimes\mu^*)\,(\sum_{i=1}^{n} a_i t^i) = \sum_i \mu^*(a_i) x^i.$$

In other words, the diagram above commutes if and only if $\mu^*(a_i) = a_i y^i$ *i.e.* if and only if a_i belongs to $A_i = (\mu^*)^{-1}(At^i)$. As for the second condition, it holds if and only if $a = \sum_i a_i$.
It now follows that $A = \bigoplus A_i$. The sum is direct because μ is an injective map of abelian groups.
Note also that $A_i A_j \subset A_{i+j}$ because μ^* is a morphism of rings. We then see that this gives an associated grading on the ring A.
Now given a grading on $A = \oplus A_i$ where we denote $a = \sum a_i$ we see that $\mu^*\colon A \to A[t,t^{-1}]$ sending $a \mapsto \sum a_i t^i$ is a ring morphism and using translations in algebra of the two conditions of a $\mathbb{G}_m$-action, we see that μ^* is indeed one. These constructions are by construction inverse to each other.

(3)(Karl) Since X is an affine scheme, let's just say $X = \mathrm{Spec}(B)$. The morphism f correspond to some morphism $\phi : B \to A$. Let ι denote the inclusion $A_0 \to A$. The pr_2 is induced by the inclusion of A into $A[t,t^{-1}]$, which we'll call i. We want to show that there is a unique $\bar{\phi}$ such that the diagram below commutes.

$$\begin{tikzcd}
A[t,t^{-1}] & & A \arrow[ll, "\mu^*"'] \\
& A_0 \arrow[ur, "\iota"] \arrow[dl, "\iota"'] & \\
A \arrow[uu, "i"] & & B \arrow[ll, "\phi"] \arrow[uu, "\phi"'] \arrow[ul, dashed, "\bar{\phi}"']
\end{tikzcd}$$

We note that for $b \in B$, $i(\phi(b)) = \mu^*(\phi(b))$ if and only if $\phi(b) \in A_0$. So we can define $\bar{\phi} = \phi^{|A_0}$ which is necessarily unique because ι is injective.