

Solutions – week 6

Exercise 1. Normal schemes and normalization

We first remark the following general fact about integral domains

$$A = \bigcap_{\mathfrak{m} \in \max(A)} A_{\mathfrak{m}}.$$

Indeed, if $x \in \bigcap_{\mathfrak{m} \in \max(A)} A_{\mathfrak{m}}$ the ideal

$$I_x = \{a \in A \mid ax \in A\}$$

needs to contain 1, implying that $x \in A$. Otherwise there is some maximal ideal $\mathfrak{m} \supset I_x$. But as we can write $x = a\lambda^{-1}$ with $a \in A$ and $\lambda \in A \setminus \mathfrak{m}$, we get that $\lambda \in I_x$ a contradiction.

(1) and (2)

Now suppose that for every maximal ideal $\mathfrak{m}$ the local ring $A_{\mathfrak{m}}$ is normal. Write $K = \text{Frac}(A)$. If $a \in K$ is the root of a monic polynomial in $A[t]$, it is therefore also the root of the same monic polynomial seen in $A_{\mathfrak{m}}[t]$, implying that $a \in A_{\mathfrak{m}}$. The above implies that $a \in A$ and as a byproduct, A is normal.

For the converse, we show that any localization of an integral normal ring is again normal. Say S is a multiplicative subset. Take $x \in K$ to be a root of a monic polynomial in $S^{-1}A[t]$. Clearing the denominators and multiplying by enough elements of S , we see that there is an $s \in S$ such that sx is a root of a monic polynomial in $A[t]$, implying that $sx \in A$, and that $x \in S^{-1}A$. (3) Note that first that if $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is dominant with A reduced, it implies that $A \rightarrow B$ is injective. Indeed, if $a \mapsto 0$, it implies that $D(a)$ does not meet the image. But then, $D(a) = \emptyset$, implying that a is nilpotent. As A is reduced, the claim follows.

Now the universal property in the category of affine schemes amounts to check equals by duality to the following. If B is normal, and $A \rightarrow B$ is injective, then there is a unique factorization $A \rightarrow \tilde{A} \rightarrow B$. Consider $K_A \rightarrow K_B$ the induced map. If $x \in K_A$ is the root of a monic polynomial in $A[t]$, the image in K_B is the root of the same polynomial seen in $B[t]$, implying that the image is in B . This concludes.

Now we prove that the universal property also holds in the category of schemes. Let $f: Z \rightarrow \text{Spec}(A)$ be any dominant map from an integral normal scheme. Cover Z by affine, necessarily normal integral, schemes (Z_i) . Then $f_i: Z_i \rightarrow \text{Spec}(A)$ factors through $\text{Spec}(\tilde{A})$ by the above. By the universal property, it glues to a necessarily unique map $\tilde{f}: Z \rightarrow \text{Spec}(\tilde{A})$.

(4)

- First we make the important remark for the construction that normalization preserves open immersions. More precisely, if $A \rightarrow A'$ is

an affine map between integral domains that induces an open immersion, then $\widetilde{A} \rightarrow \widetilde{A}'$ also induces an open immersion. The key is that if S is a multiplicative subset of A , we have $\widetilde{S^{-1}A} = S^{-1}\widetilde{A}$. Using this we show the claim. That $A \rightarrow A'$ is an open immersion means that there exists a finite number of functions $a_1, \dots, a_n \in A$ such that the localization $A_{a_i} \rightarrow A'_{a_i}$ is an isomorphism, and the image of the a_i 's generated the unit ideal in A' . Using that we can commute localization and normalization as stated above, we get that the maps $\widetilde{A}_{a_i} \rightarrow \widetilde{A}'_{a_i}$ are also isomorphisms also, showing the claim.

- Now we show that any separated integral scheme admits a normalization. Say X is such a scheme, and that X is covered by affine schemes X_i 's with affine intersection (by separated) X_{ij} . We claim that we can glue the schemes $\widetilde{X}_i$'s to a scheme $\widetilde{X}$ together with a map $\widetilde{X} \rightarrow X$. By the above the image of $\varphi_{ij,i}: \widetilde{X}_{ij} \rightarrow \widetilde{X}_i$ is open. We write it $U_{i,j}$. Now note that $\varphi_{ij,j}\varphi_{ij,i}^{-1}: U_{i,j} \rightarrow U_{j,i}$ is an isomorphism. We denote this last map $\psi_{i,j}$. Using the universal property of $\widetilde{}$ in the affine case, it follows that $(\psi_{i,j})_{(i,j)}$ is a collection that satisfies the cocycle condition, allowing us to proceed to the usual gluing construction. Note that the maps $\widetilde{X}_i \rightarrow X_i$ glue by construction to a map $\widetilde{X} \rightarrow X$. To show that this has the required universal property, let $f: Z \rightarrow X$ be any dominant map from an integral normal scheme. Write $Z_i = f^{-1}(X_i)$. By the above affine case there is a unique map $\widetilde{f}: Z_i \rightarrow \widetilde{X}_i \rightarrow \widetilde{X}$ which glues to a necessarily unique map $\widetilde{f}: Z \rightarrow \widetilde{X}$ showing the claim.
- The general case follows by the same pattern and the separated case. Namely, any scheme can be covered by an union of open separated (affine) schemes such that the intersection is separated.

Exercise 2. This exercise was a previous year hand-in exercise so solutions are credited to past students of the course who wrote them.

(1)(Joel) Let $A = k[x, y]$, $I = (x^2, y^2)$ and $R = A/I$. Consider the map $\phi: R[Z, W] \rightarrow \bigoplus_{n \geq 0} I^n$ which sends $Z \rightarrow x^2$ and $W \rightarrow y^2$ in degree one. Then $\ker \phi = (Zy - Wx)$, so $\text{Bl}_I \cong \text{Proj } R[Z, W]/(Zy^2 - Wx^2)$. Next, we show that the blow-up is not normal. Consider the affine chart U_W where $W \neq 0$, which is given by $\text{Spec } k[x, y, z]/(zy^2 - x^2) =: \text{Spec } A$, where $z = \frac{Z}{W}$. Then $\frac{zy}{x} \in \text{Frac}(A)$, and $(\frac{zy}{x})^2 = \frac{z \cdot zy^2}{x^2} = \frac{zx^2}{x^2} = z$. Hence, $\frac{zy}{x}$ is a root of the monic polynomial $P(t) = t^2 - z$ with coefficients in A . Now $\frac{zy}{x} \notin A_{((x,y))}$, as x is not inverted in the localization, and the field of fractions of $A_{((x,y))}$ is the same as for A , we see that the blow-up is not normal.

The affine chart U_W can also be expressed as $\text{Spec } k[x, y, \frac{x^2}{y^2}]$, and similarly we get a chart $U_Z \cong \text{Spec } k[x, y, \frac{y^2}{x^2}]$. As above, neither of these affine charts are normal, and we can normalize on the ring level by exercise (2). Hence, for $k[x, y, \frac{x^2}{y^2}]$ the normalization is given by $k[x, y, \frac{x^2}{y^2}][\frac{x}{y}] = k[x, y, \frac{x}{y}] = k[y, \frac{x}{y}] =$

$k[y, t]$, and similarly for we get $k[x, t']$ as the normalization for the ring corresponding to U_Z . Thus we have two affine planes over k as the normalizations of our charts.

Let us inspect the blow-up of $J = (x, y)$: the blow-up algebra Bl_J is isomorphic to $\tilde{A} := k[x, y][Z, W]/(xW - yZ)$ by the same procedure as in the beginning. Here, we have the charts $\tilde{U}_W = \text{Spec } k[x, y, z']/(x - yz') \cong \text{Spec } k[x, y, \frac{x}{y}] = \text{Spec } k[y, \frac{x}{y}] = k[y, t]$ when $W \neq 0$ and similarly $\tilde{U}_Z = \text{Spec } k[x, y, w']/(x - yw') \cong \text{Spec } k[x, t']$, which are the normalizations of the two affine charts of the blow up of (x^2, y^2) . Now, on the intersection $U_Z \cap U_W$ of $\text{Proj } \text{Bl}_I$ we have $Z, W \neq 0$, so $U_Z \cap U_W \cong k[x, y, \frac{x^2}{y^2}, \frac{y^2}{x^2}]$, with its normalization given by $k[x, y, \frac{x}{y}, \frac{y}{x}] = k[y, \frac{x}{y}, \frac{y}{x}]$, which corresponds to the intersection on the blow up of (x, y) . Hence, we can glue to get the normalization of the blow-up of (x^2, y^2) .

(2)(Joel) Let $I = (x^2, y)$ and $R = A/I$. The blow-up is isomorphic to $\text{Bl}_I \cong \text{Proj } A[Z, W] = \text{Proj } k[x, y][Z, W]/(yZ - x^2W)$, which we can cover with $U_Z = \text{Spec } k[x, y][a]/(y - ax^2)$ and $U_W = \text{Spec } k[x, y][b]/(by - x^2)$, where $a = \frac{W}{Z}$ and $b = \frac{Z}{W}$. On U_W we see that at the point $x = b = 0$ the scheme is not regular by the criterion provided in the exercise, as $(0, 0, 0) \in V(by - x^2, -2x, b, y)$. This is the only non-regular point, as on U_Z we have $V(y - ax^2, -2a, 1, -x^2) = \emptyset$. Hence all points except $x = y = W = 0$ are regular.

(3)(Julie) Let $g = xy - zw \in k[x, y, z, w]$. By the criterion provided in the statement of the exercise, the set of non-regular points in

$$\text{Spec } \left(\frac{k[x, y, z, w]}{(xy - zw)} \right)$$

is given by

$$V(g, \partial_x g, \partial_y g, \partial_z g, \partial_w g) = V(xy - zw, y, x, -w, -z) = V(x, y, z, w) = \{(x, y, z, w)\},$$

where the last equality holds by maximality of (x, y, z, w) in $k[x, y, z, w]$. Hence, all points of X are regular except for (x, y, z, w) (corresponding to the origin in $V(xy - zw) \subseteq \mathbb{A}_k^4$).

(4)(Maxence) Consider $R = k[x, y, z, w]$. Let $I = (x, y, z, w)$, $I' = (x, z)$ and $J = (xy - zw)$. We consider the strict transform St_J (resp. St'_J) of $V(J) = X$ at I (resp. I') in $\mathbb{A}_k^4$. We denote these schemes as respectively X_1 and X_2 . We know that X_1 (resp. X_2) is the closed subscheme $V_+(\bigoplus_n I^n \cap J)$ of Bl_I (resp. the closed subscheme $V_+(\bigoplus_n I'^n \cap J)$ of $\text{Bl}_{I'}$).

Notice that $\text{Bl}_I = \text{Proj}(R[X, Y, Z, W]/\tilde{I})$ and $\text{Bl}_{I'} = \text{Proj}(R[X, Z]/\tilde{I}')$ where

$$\tilde{I} = (yX - xY, zX - xZ, wX - xW, yZ - zY, yW - wY, zW - wZ) \text{ and } \tilde{I}' = (zX - xZ).$$

So, the preimage of the ideal $\bigoplus_n I^n \cap J$ by the natural surjection is given by the ideal $K = \tilde{I} + (xy - zw, xY - zW, XY - ZW)$. Indeed, it must be generated in $R[X, Y, Z, W]$ by homogeneous polynomials with degree less or equal to 2 with respect to the variables X, Y, Z, W whose image is sent to the generator of J which has degree 2. These generators are enough since every elements f in I^n has monomials of at least degree n and if $f \in J$, then

$f = g \cdot (xy - zw)$. Since $xy - zw$ is of degree 2, the polynomial g must be of degree $n - 2$, hence $g \in I^{n-2}$. So for every element in $I^n \cap J$ with $n \geq 3$ can be reached using generators of K . In the same way the preimage of $\bigoplus_n I^n \cap J$ by the natural surjection is the ideal $K' = \tilde{I}' + (xy - zw, yX - wZ)$.

That is,

$$X_1 = \text{Proj}(R[X, Y, Z, W]/K) \text{ and } X_2 = \text{Proj}(R[X, Z]/K').$$

For X_1 on $D_+(X)$, we have $\mathcal{O}_{X_1}(D_+(X)) = k[x, s_1, s_2, s_3]/(s_1 - s_2s_3)$ by simplifying the equations of K . And by the criterion, the affine open subset $D_+(X)$ of X_1 is regular. The same result holds for $D_+(Y), D_+(Z)$ and $D_+(W)$ by symmetry of the variables. Hence X_1 is regular.

For X_2 on $D_+(X)$, we have $\mathcal{O}_{X_2}(D_+(X)) = k[x, w, s]$ by simplifying equations of K' , and so $D_+(X) = \mathbb{A}_k^3$ which is regular. The same result holds for $D_+(Z)$ by symmetry. Hence X_2 is regular.

(5)(Maxence) We want to compute the fiber of $f_1 : X_1 \rightarrow X$ and $f_2 : X_2 \rightarrow X$ over (x, y, z, w) .

First, the residue field of $(x, y, z, w) \in X$ is simply k by exactness of localization, so for $i = 1, 2$, we need to compute the fibred product $X_i \times_X \text{Spec}(k)$. Hence, if we denote $A = k[x, y, z, w](xy - zw)$ we have

$$X_1 \times_X \text{Spec}(k) = \text{Proj} \left(\frac{A[X, Y, Z, W]}{K} \otimes_A k \right) \text{ and } X_2 \times_X \text{Spec}(k) = \text{Proj} \left(\frac{A[X, Z]'}{K} \otimes_A k \right).$$

Looking at these tensor products, by using A -linearity all relations given by K vanish except $XY - ZW = 0$ in the residue field of (x, y, z, w) by its definition. The same holds for K' but here all its relations vanish. It yields that

$$X_1 \times_X \text{Spec}(k) = \text{Proj}(k[X, Y, Z, W]/(XY - ZW)) = \mathbb{P}_k^1 \times_{\text{Spec}(k)} \mathbb{P}_k^1$$

and

$$X_2 \times_X \text{Spec}(k) = \text{Proj}(k[X, Z]) = \mathbb{P}_k^1.$$

Exercise 3. If B is reduced any localization is also. Therefore the degree zero part of any localization by homogeneous elements are also. It implies that $\text{Proj}(B)$ is reduced. If B is integral, the product ss' of two non-zero homogeneous elements s, s' is never zero. It implies that the degree zero part of $B_{ss'}$ is not zero also. It implies that the intersection of two non-empty opens is never empty in $\text{Proj}(B)$. Therefore $\text{Proj}(B)$ is irreducible. Being also reduced, it is integral.

Exercise 4. (1) The fiber over 2 is not reduced. The fiber over 3 is reduced but not integral. It is integral over any other prime by Eisenstein criterion.

(2) The fiber over p is not reduced and not irreducible. Otherwise it is isomorphic to $k[x, y, y^{-1}]$ where k is a prime field not equal to $\mathbb{F}_p$.

Exercise 5. Statements (1) and (2) are true (proof below), for (3) take $\text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(\mathbb{R})$ while a counter example to the remaining is the map $\text{Spec}(\mathbb{F}_p(t^{1/p})) \rightarrow \text{Spec}(\mathbb{F}_p(t))$, base changed against itself.

Let us start with open immersions. Up to composing by an isomorphism we can suppose that $f: X \rightarrow Y$ is $U \subset Y$ an open.

But now we see that the following is a pullback diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

Let $U' = g^{-1}(U) \subset Y'$ open, equipped with the open-subscheme of Y' -structure. Indeed the universal property of the pullback here reads as a map $Z \rightarrow Y'$ that topologically factors to the open $f^{-1}(U)$, implying that it factors schematically because the sheaf on the open is just the restriction of the sheaf on the all set.

We now prove and (2). First, a map $f: X \rightarrow Y$ is a closed immersion if and only if $f: f^{-1}(U_i) \rightarrow U_i$ is a closed immersion for $\bigcup U_i = Y$ an open cover. Indeed a subset $Z \subset Y$ is closed if and only if $U_i \cap Z \subset U_i$ is closed for every i and to check that the desired sheaf map is surjective is and only if it is locally.

Therefore if

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a pullback diagram with f being a closed immersion, we can reduce to the affine case as follows. First, take a cover (U_i) of Y by affines, and consider the cover of X induced by the pre-images $(g^{-1}(U_i))$. Then cover each of these opens $g^{-1}(U_i)$ by affines (V_{ij}) . Then

$$\begin{array}{ccc} f'^{-1}(V_{ij}) & \longrightarrow & f^{-1}(U_i) \\ f' \downarrow & & \downarrow f \\ V_{ij} & \longrightarrow & U_i \end{array}$$

is again a pullback diagram.

We now use the following lemma.

Lemma. *Let $X = \text{Spec}(A)$ be affine and $\iota: Z \rightarrow X$ a closed immersion. Then the natural map $Z \rightarrow \text{Spec}(\mathcal{O}_Z(Z))$ is an isomorphism and*

$$A \rightarrow \mathcal{O}_Z(Z)$$

is surjective. If I is the kernel of this map, we therefore have

$$\begin{array}{ccc} Z & \xrightarrow{\iota} & \text{Spec}(A) \\ \sim \downarrow & \nearrow & \\ \text{Spec}(A/I) & & \end{array}$$

Proof. Let $Z = \cup_i V_i$ a finite covering by affines. By hypothesis $V_i = U_i \cap Z$ for some open U_i of X . Covering all U_i and $X \setminus Z$ by finitely many principal opens of X we can suppose that $V_i = D(f_i) \cap Z$ for some $f_i \in \mathcal{O}_X(X)$ with (f_i) being the unit ideal in A , and therefore in $\mathcal{O}_Z(Z)$ also. Now we use week 5.5.2 to conclude that Z is affine. Therefore $Z \rightarrow \text{Spec}(\mathcal{O}_Z(Z))$ is an isomorphism.

By assumption for every $\mathfrak{p} \in \text{Spec}(A)$ the map $\mathcal{O}_{X,\mathfrak{p}} \rightarrow (\iota_* \mathcal{O}_Z)_{\mathfrak{p}}$ is surjective. When $\mathfrak{p} \notin Z$ the right is zero and coincides with $\mathcal{O}_Z(Z)_{\mathfrak{p}}$: indeed take $\mathfrak{p} \in D(f_i) \subset X \setminus Z$, then as $D(f_i) \cap Z = \emptyset$, we conclude that f_i in $\mathcal{O}_Z(Z)$ is nilpotent and as $\mathcal{O}_Z(Z)_{\mathfrak{p}}$ is a further localization of $\mathcal{O}_Z(Z)_{f_i} = 0$ we have our claim. When $\mathfrak{p} \in Z$ the right hand side is $\mathcal{O}_{Z,\mathfrak{p}}$, and because X and Z are affine this is $A_{\mathfrak{p}} \rightarrow \mathcal{O}_Z(Z)_{\mathfrak{p}}$. So we conclude that $A \rightarrow \mathcal{O}_Z(Z)$ is a map of A -modules surjective at every localization at primes, implying that this map is surjective. $\square$

Therefore $f^{-1}(U_i)$ is also affine. Because the inclusion of affine schemes into schemes preserve limits, we are therefore in a situation

$$\begin{array}{ccc} \text{Spec}(B \otimes_A A/I = B/IB) & \longrightarrow & \text{Spec}(A/I) \\ \downarrow f' & & \downarrow f \\ \text{Spec}(B) & \xrightarrow{g} & \text{Spec}(A) \end{array}$$

which concludes.

Exercise 6. We use the cover $D(x) \cup D(y)$ and the sheaf property to compute global sections of U . Because x, y are non zero divisors, localization maps $R[x^{\pm 1}, y] \rightarrow R[x^{\pm 1}, y^{\pm 1}]$ and $R[x, y^{\pm 1}] \rightarrow R[x^{\pm 1}, y^{\pm 1}]$ are injective and we may treat them as inclusions. Now, global sections are the elements of the kernel of the map

$$R[x^{\pm 1}, y] \times R[x, y^{\pm 1}] \rightarrow R[x^{\pm 1}, y^{\pm 1}]$$

that sends $(f, g) \mapsto f - g$. In other words

$$\mathcal{O}(U) = R[x^{\pm 1}, y] \cap R[x, y^{\pm 1}] = R[x, y].$$

If U was affine, then the natural $U \rightarrow \text{Spec}(R[x, y])$ would be an isomorphism, because it is an inclusion of an open, an equality. But because $R \neq 0$, $\text{Spec}(R[x, y] \setminus U) = \text{Spec}(R)$ is non empty, a contradiction.