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AG II - Schemes and sheaves

Solutions – week 4

Exercise 1. *Tangent vectors.*

(1) We have $d(1) = d(1^2) = 2d(1)$ implying that $d(1) = 0$. Now using R-linearity $d(r) = rd(1) = 0$ for any $r \in R$.

(2) The kernel of the projection is $0 \oplus N$. Note that $(0, n) \cdot (0, n') = (0, 0)$ implying the square zero requirement.

(3) Immediate from the definition of the law.

(4) The k-algebra morphism $k[t] \to k \oplus_0 k$ sending t to $(0, 1)$ factors through $k[t]/t^2$, and is surjective. By equality of dimensions, because this map is a map of finite dimensional k-vector spaces, we deduce that this map is an isomorphism.

(5) First we note that as x is a k-rational point we have a k-algebra section of the surjection $\mathcal{O}_{X,x} \to k(x)$. Using this, we may write

$$\mathcal{O}_{X,x} = k(x) \oplus \mathfrak{m}_x$$

as a direct sum of k-vector spaces. Note that a k-derivation $d\colon \mathcal{O}_{X,x} \to k(x)$ will have to send the first component to zero by the first point of the exercise. Note also that if $f, g \in \mathfrak{m}_x$ then $d(fg) = f(x)d(g) + g(x)d(f) = 0$, because we have $f(x) = g(x) = 0$. Therefore, any derivation necessarily factors through $\mathfrak{m}_x/\mathfrak{m}_x^2$. It is therefore also sufficient for a k-derivation $\mathcal{O}_{X,x} \to k(x)$ to define a k-linear map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k(x)$. This shows the first isomorphism.

For the last one, note that a k-scheme morphism from $\mathrm{Spec}(k[\epsilon])$ to X sending the point to x is equivalent to the data of a local k-algebra map $\mathcal{O}_{X,x} \to k[\epsilon]$. Projecting to $k\epsilon$ and then using the identification $k \cong k(x)$, it defines a derivation $\mathcal{O}_{X,x} \to k(x)$. The other way around, given $d\colon \mathcal{O}_{X,x} \to k(x)$, we define a k-algebra map $\mathcal{O}_{X,x} \to k[\epsilon]$ by (ev_x, d).

Exercise 2. (2) Let $k \to l$ be a Galois extension. Let V be a k-vector space. We consider the base change $V_l = V \otimes_k l$. We consider the Galois action of $G = \mathrm{Gal}(l\colon k)$ on V_l given by $g \cdot (v \otimes \lambda) = v \otimes g(\lambda)$.

The goal is to show that $(V_l)^G = V$, more precisely the image of $V \to V \otimes_k l$ by $v \mapsto v \otimes 1$. We may sometimes write V for this image in what follows.

Suppose first that V is finite dimensional. Write $V = \bigoplus_i V_i$ with V_i being one dimensional. Then

$$V_l = \oplus V_{i,l} = \prod V_{i,l}.$$

and the action G restricts to this direct sum/product.

If V is one dimensional, then $(V\otimes_k l) = V$ follows from Galois theory. Note that we can compute the fixed points of a direct product as the fixed points of each components. Therefore, the case where V is finite dimensional follows: indeed $V \subset V_l^G$ with the same dimension.

To treat the arbitrary case, we want to show $(V \otimes_k l)^G \subset V$. Let $v \in (V \otimes_k l)^G$. Then there exists a finite dimensional subspace $W \subset V$ such that v is in W_l. Therefore we see by the previous case that $v \in W \subset V$, which conclude the argument.

Exercise 4. *Properties of maps* We show that if $\varphi\colon A \to B$ is injective, then the topological image of $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is dense. Take any $a \in A$ not nilpotent. Because the map is injective $\varphi(a) \in B$ is also not nilpotent. Therefore $f^{-1}(D(a)) = D(\varphi(a)) \neq \emptyset$, implying that $D(a)$ intersects with the image of $\operatorname{Spec}(B)$, showing the claim.

Exercise 5. *Functoriality of* Proj*.*

(2) For example $\iota\colon \mathbb{Z}[x,y] \to \mathbb{Z}[x,y,z]$ the inclusion. Then $U(\iota) = D_+(x) \cup D_+(y)$. In particular $(X,Y) = [0:0:1]$ is not in this open.

(3) We use that $U(\psi)$ is a gluing

$$U(\psi) = \bigcup_{a\in A_+ \text{ hom.}} \operatorname{Spec}(B_{(\psi(a))})$$

with gluing data is given by, at the dual level of ring of functions by the natural maps

$$\begin{tikzcd} B_{(\psi(a))} \arrow[dr] & \\ B_{(\psi(a'))} \arrow[r] & B_{(\psi(aa'))} \end{tikzcd}$$

So to define a map out of $U(\psi)$, it suffices to define a map from each $\operatorname{Spec}(B_{(\psi(a))})$. To do this we glue the map induced by ψ

$$\operatorname{Spec}(B_{(\psi(a))}) \to \operatorname{Spec}(A_{(a)}).$$

This glues because the following commutes as every map is giving by the natural extension-restriction of ψ to these rings

$$\begin{tikzcd} A_{(a)} \arrow[r] \arrow[dr] & B_{(\psi(a))} \arrow[dr] & \\ & A_{(aa')} \arrow[r] & B_{(\psi(aa'))} \\ A_{(a')} \arrow[r] \arrow[ur] & B_{(\psi(a'))} \arrow[ur] & \end{tikzcd}$$

(4) We first show that $U(\psi) = \mathrm{Proj}(B)$. Let $\mathfrak{p} \in D_+(b)$ any point of $\mathrm{Proj}(B)$ and some homegeneous $b \in B_+$. Note that b^{dk_0} is in the image of ψ by hypothesis, which concludes.

To show that the map is a closed immersion, it suffices to show that locally

$$\mathrm{Spec}(B_{(\psi(a))}) \to \mathrm{Spec}(A_{(a)})$$

is a closed immersion. But therefore it suffices to show that the underlying map of rings is a surjection. Because $D_+(a) = D_+(a^N)$ for any $N \geq 1$ we can suppose that $\deg(a) \geq k_0$. But then an element of $B_{(\psi(a))}$ is of the form $\frac{b}{\psi(a)^n}$ with $\deg(b) = dn \deg(a)$. But as $A_{n\deg(a)} \to B_{dn\deg(a)}$ is surjective by assumption, we win.

Now we are left to show that the image is $V_+(\ker(\psi))$. It suffices to show that it's the image when intersecting to every $D_+(a)$. But $D_+(a) \cap V_+(\ker(\psi)) = V(\ker(\psi_{(a)}))$, which concludes.

(6) Same local trick. It suffices to show that it's locally an isomorphism. Enlarging degrees is again harmless.