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AG II - Schemes and sheaves

Solutions – week 12

Exercise 1. *Homotopy invariance of class groups.* Let X be integral, Noetherian, separated and regular in codimension 1.

(1) Show that $X \times \mathbb{A}^1$ is also integral, Noetherian, separated and regular in codimension 1.
(2) Show that the projection π to the first component induces a morphism $\pi^*\colon \operatorname{Cl}(X) \to \operatorname{Cl}(X \times \mathbb{A}^1)$.
(3) Show that π^* is an isomorphism.

You may look at II.6.6 in Hartshorne.

Exercise 2. *A Künneth formula for class groups.* Let X be an integral, separated, Noetherian and locally factorial scheme. Let $n \geq 1$. Show that $\mathbb{P}^n_X = X \times \mathbb{P}^n_{\mathbb{Z}}$ also satisfies the above and that

$$\operatorname{Cl}(X \times \mathbb{P}^n_{\mathbb{Z}}) \cong \operatorname{Cl}(X) \times \mathbb{Z}.$$

Hint: Consider $\phi\colon \mathbb{P}^n_{K(X)} \to X \times \mathbb{P}^n_{\mathbb{Z}}$*. Show that* $\phi^*\colon \operatorname{Pic}(\mathbb{P}^n_X) \to \operatorname{Pic}(\mathbb{P}^n_{K(X)}) \cong \mathbb{Z}$ *gives a retraction of the first arrow in the exact sequence (exercise 8, week 9)*

$$\mathbb{Z} \to \operatorname{Cl}(\mathbb{P}^n_X) \to \operatorname{Cl}(\mathbb{A}^n_X) \to 0$$

coming from the divisor $V_+(X_0)$ *in* $\operatorname{Cl}(\mathbb{P}^n_X)$.

Exercise 3. *Very ample divisors.* Let k be a field. Let S be a $\mathbb{N}$-graded ring finitely generated in degree 1 with $S_0 = k$. Denote by $X = \operatorname{Proj}(S)$. Suppose that X is integral and $\mathcal{O}_X(X) = k$.[1]

(1) Show that $\mathcal{O}_X(1)$ is k-very ample.
(2) If $\dim(X) \geq 1$, show that $\mathbb{Z} \xrightarrow{\mathcal{O}_X(1)} \operatorname{Pic}(X)$ is injective.
Hint: If $\mathcal{O}_X(1)$ *is torsion, it would imply that* $\mathcal{O}_X$ *is* k*-very ample.*
(3) If X is normal, deduce that if $0 \neq s \in \mathcal{O}_X(1)(X)$, then $\operatorname{div}(s) \in \operatorname{Cl}(X)$ has infinite order.

Solution key. (1) Denote by $s_0, \dots, s_n$ degree 1 elements that $S_0 = k$ generates S as an algebra. Note that $k[x_0, \dots, x_n] \to S$ sending $x_i \mapsto s_i$ is a graded surjection and therefore induces a closed immersion $\iota\colon X = \operatorname{Proj}(S) \to \mathbb{P}^n_k$. Note that by construction $\iota^*\mathcal{O}_{\mathbb{P}^n_k}(1) \cong \mathcal{O}_X(1)$. The claim now follows.
(2) If $\mathcal{O}_X(1)$ is torsion, meaning that $\mathcal{O}_X(n) \cong \mathcal{O}_X$, it would mean that $\mathcal{O}_X$ is k-very ample. As we suppose that $\mathcal{O}_X(X) = k$, this

[1] This condition follows from previous assumptions if k is algebraically closed.

would imply that X is a point, a contradiction with the dimension hypothesis.

(3) Follows from the injection $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ (normal) and the previous point.

$\square$

Exercise 4. *Projective Cone.* This exercise is a follow-up to exercise 2, week 7.

Let S be a $\mathbb{N}$-graded ring finitely generated in degree 1 over S_0. Consider the $\mathbb{N}$-graded ring $S[t]$ with elements of S keeping their grading and with t placed in degree 1. We call $\mathrm{Proj}(S[t])$ with this grading the *projective cone.*

(1) Show that this grading comes from the product action

$$\mathbb{G}_{m,S_0} \times_{S_0} \mathrm{Spec}(S) \times_{S_0} \mathbb{A}^1_{S_0} \xrightarrow{(\mu_S \,\mathrm{pr}_{12}, \mu_{\mathbb{A}^1} \mathrm{pr}_{13})} \mathrm{Spec}(S) \times_{S_0} \mathbb{A}^1_{S_0}$$

where pr_{ij} denote projections, μ_S the action on $\mathrm{Spec}(S)$ and $\mu_{\mathbb{A}^1}$ the usual $\mathbb{G}_m$-action on $\mathbb{A}^1$.

(2) Show that there are natural identifications $V_+(t) = \mathrm{Proj}(S)$ and $D_+(t) = \mathrm{Spec}(S)$. Show furthermore that $V_+(S_+)$ (taken in $\mathrm{Proj}(S[t])$) identifies to the vertex (see exercise 6, week 10) in $\mathrm{Spec}(S)$. We therefore denote this closed subscheme by v.

(3) Let $s_0, \dots, s_n$ be generators of S in degree 1. Show that $\mathrm{Proj}(S[t]) \setminus v$ is covered by the open sets $D_+(s_i)$ and that each open set is isomorphic to $\mathrm{Spec}(S_{(s_i)}[t])$. Deduce that we have a natural map

$$p\colon \mathrm{Proj}(S[t]) \setminus v \to \mathrm{Proj}(S).$$

(4) Let k be an algebraically closed field, and suppose $S_0 = k$. Suppose that S is integral, Noetherian and normal. Suppose that $X = \mathrm{Proj}(S)$ is of dimension ≥ 1. Show that p^* induces an isomorphism on class groups. Deduce that, if $C = \mathrm{Spec}(S)$ denotes the cone of X then we have an exact sequence

$$1 \to \mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(C) \to 1$$

where the first morphism sends 1 to the class of $\mathcal{O}_X(1)$, and the second is the composition of p^* and the restriction to C.

Proof. (1) Analyze that degree 1 elements are still of degree 1.

(2) The $V_+(t)$ asserion is immediate. For the $D_+(t)$ it amounts to realizing that degree zero elements of S_t are just S. For the last assertion, note that $\mathrm{Proj}(S_0[t])$ identifies with $\mathrm{Spec}(S_0)$.

(3) Note that $(s_0, \dots, s_n)$ generates S_+ as an ideal. Therefore the claim on the cover follows. Note that the degree zero part of $S[t]_{s_i}$ identifies to $S_{(s_i)}[\frac{t}{s_i}]$ which gives the claim.

(4) We denote by $\overline{X}$ the projective cone.

We show that p^* induces an isomorphism on class groups. The previous point shows that $K(\overline{X}) = K(X)(T)$ for $T = t/s_i$ for some i. From this observation, one can actually apply the same proof as proposition chapter 2, 6.6 in Hartshorne. We give a bit more details. We use the terminology this proof in what follows. We can separate the codimension points of $\overline{X} \setminus v$ in two distinct families. Type 1 points will be points such that the image in X is of codimension 1. Type 2 are points such that the image is generic. These points are in one to one correspondence with codimension points of $\mathbb{A}^1_{K(X)}$ along the dominant map $\mathbb{A}^1_{K(X)} \to \overline{X} \setminus v$. We can define π^* exactly as in proposition 6.6 at the level of divisors, and it gives a surjection to the subgroup of $\mathrm{Cl}(\overline{X} \setminus v)$ generated by type 1 points. Note that the exact same proof as the one in proposition 6.6 shows that any type 2 points are linearly equivalent to type 1 points. This shows surjectivity. Injectivity also is proven with the exact same argument.

Note also that by exercise 2.3 and the exercise 4, week 10 we have an exact sequence (where $1 \in \mathbb{Z}$ is sent to $\mathcal{O}_X(1)$ seen as $V_+(t)$, which is prime because X is integral)

$$1 \to \mathbb{Z} \to \mathrm{Cl}(\overline{X}) \to \mathrm{Cl}(C) \to 1$$

because as argued above $D_+(t) = \mathrm{Spec}(S) = C$.

Note that because $v \in \overline{X}$ is at least of codimension 2, we have $\mathrm{Cl}(\overline{X}) \cong \mathrm{Cl}(\overline{X} \setminus v)$. We can now use that p^* induces an isomorphism to conclude.

□

Exercise 5. *Computations of class groups on quadric hypersurfaces.* Suppose that k is algebraically closed and $char(k) \neq 2$. Let $2 \leq r \leq n$. Consider the ring (equipped with the standard grading)

$$S_r = k[x_0, \dots, x_n]/(x_0^2 + \cdots + x_r^2).$$

You can assume that this ring is normal. (See Hartshorne, exercise 6.4 for a proof).

(1) Show that up to a linear change of variable, we can suppose that

$$S_r = k[x_0, \dots, x_n]/(x_0x_1 + x_2^2 \cdots + x_r^2).$$

Denote by $C_r = \mathrm{Spec}(S_r)$ and $X_r = \mathrm{Proj}(S_r)$.

(2) Show that $\mathrm{Cl}(C_r)$ is cyclic when $r \neq 3$
Hint: Consider the prime divisor $V(\sqrt{(x_1)})$ and the exact sequence of week 9, exercise 8.

(3) Show that $\mathrm{Cl}(C_2) \cong \mathbb{Z}/2\mathbb{Z}$.
Hint: Consider the same exact sequence. See Hartshorne example 6.5.2.

(4) Show that $\mathrm{Cl}(C_3) \cong \mathbb{Z}$.
Hint: show that after a suitable change of variable we see that $X_r \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$. Then use exercise 1 and the exact sequence of the last point of the above exercise.

(5) Show that $\mathrm{Cl}(C_r) \cong 0$ of $r \geq 4$. In particular, S_r is factorial.
Hint: show that (x_1) is prime in this case and conclude.

(6) Use the exact sequence of the last point of the above exercise to compute $\mathrm{Cl}(X_r)$ for all $r \geq 2$.

Solution key. (1) Note that $x_0^2 + x_1^2 = (x_0 + ix_1)(x_0 - ix_1)$ where $i \in k$ is a root of -1 in k. Because $char(k) \neq 2$ we can set $y_0 = x_0 + ix_1$ and $y_1 = x_0 - ix_1$ two different variables.

(2) If $r \neq 3$ note that $V(x_1)$ is irreducible. We can then use the same strategy as in example 6.5.2 in Hartshorne.

(3) See example 6.5.2 in Hartshorne.

(4) Up to a change of variable we recognize the Segre embedding of $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$ in $\mathbb{P}^3_k$. We may work with

$$S_r = k[x_0, x_1, x_2, x_3]/(x_0x_1 - x_2x_3)$$

Note that in the exact sequence from the previous exercise, $1 \in \mathbb{Z}$ is sent to the Cartier corresponding to $p_1^*\mathcal{O}_(1) \otimes p_2^*\mathcal{O}(1)$ when we view this in the product. In other words, this is the class of $(1,1) \in \mathrm{Cl}(\mathbb{P}^1_k \times_k \mathbb{P}^1_k) = \mathbb{Z} \oplus \mathbb{Z}$. The result follows.

(5) $V(x_1)$ is principal and prime.

(6) Follows from the exact sequence.

□

Exercise to hand in. *Morphisms between projective spaces, again.* (Due 16 December, 18:00) Please write your solution in TEX.
Let k be a field. You may find part (1), (2) of the exercise useful while solving part (3) of the exercise.

(1) (**Graded Nakayama**) Let $R = \underset{n\in\mathbb{N}}{\oplus} R_n$ be a graded ring and $M = \underset{n\in\mathbb{N}}{\oplus} M_n$ be a graded R module, $J \subseteq R_+$ be a homogeneous ideal of R. If $M = JM$, then show that $M = 0$. Recall that R_+ is the homogeneous ideal $\underset{n>0}{\oplus} R_n$ of R.

(2) Let R be a Noetherian ring generated in degree 1 as an R_0-algebra.[2] Let $F_0, \dots, F_m$ be homogeneous elements of strictly positive degrees of R. Assume that the radical of $(F_0, \dots, F_r)$ is R_+. Show that the graded ring inclusion $R_0[F_0, \dots, F_r] \to R$ is a finite ring map. You may want to use part (1).

(3) Given a morphism of k-schemes $f\colon \mathbb{P}^n_k \to \mathbb{P}^m_k$, show that the image is either a point; and if the image is not a point, then $m \geq n$ and the image has dimension n. In the second case, show that the morphism is finite. *Hint: we have* $f^*\mathcal{O}_{\mathbb{P}^m_k}(1) \cong \mathcal{O}_{\mathbb{P}^n_k}(d)$ *for some* $d \geq 0$. *Break the study into two cases:* $d = 0$ *and* $d \geq 1$. *In this last case show that the polynomials* $F_0, \dots, F_m$ *homogeneous of degree* d *which defines the map generates an ideal which radical is* $k[X_0, \dots, X_n]_+$.

(4) Identify $\mathbb{P}^n_k = \mathrm{Proj}(k[x_0, \dots, x_n])$. Show that the projection map from $\mathbb{P}^n_k - [0 : 0 : \dots : 1]$ to $\mathbb{P}^{n-1}_k$ given by the sections $x_0, x_1, \dots x_{n-1} \in \mathcal{O}_{\mathbb{P}^n_k}(1)$ cannot be extended to $\mathbb{P}^n_k$.

Remark. When the image of a map $\mathbb{P}^n_k \to \mathbb{P}^m_k$ is not a point, the map $\mathbb{P}^n_k \to \mathbb{P}^m_k$ can be written as a composition of (1) a Veronese embedding

[2]It is enough to assume that R is $\mathbb{N}$-graded Noetherian.

$\mathbb{P}_k^n \to \mathbb{P}_k^N$, for some N, (2) an automorphism of $\mathbb{P}_k^N$, (3) a projection map

$$\mathbb{P}_k^N - V(X_0, \ldots, X_{m'}) \to \mathbb{P}_k^{m'},$$

sending $(x_0 : \ldots : x_N)$ to $(x_0 : \ldots : x_{m'})$, for some $m' \leq m$, (4) a linear embedding $\mathbb{P}_k^{m'} \to \mathbb{P}_k^m$ – here, a linear embedding is map that sends $\underline{x} = (x_0 : \ldots : x_{m'})$ to $(x_0 : \ldots : x_{m'} : L_{m'+1}(\underline{x}) : \ldots : L_m(\underline{x}))$, where the L_j's are some linear polynomials in $m' + 1$ variable with k-coefficients – and (5) an automorphism of which is a permutation of variables $\mathbb{P}_k^m$.
Indeed, a map $\mathbb{P}_k^n \to \mathbb{P}_k^m$ is given by homogeneous $m + 1$ polynomials of some fixed degree d, that we denote by $F_0, \ldots, F_m$. By functoriality of Proj we can express this map (where the first map is given by $X_i \mapsto F_i$)

$$k[X_0, \ldots, X_m] \to (k[X_0, \ldots, X_n])_d \subset k[X_0, \ldots, X_n].$$

So we can factor by a Veronese embedding

$$k[X_0, \ldots, X_m] \to k[Y_j]_{j=0}^N \to (k[X_0, \ldots, X_n])_d$$

sending X_i to lifts of F_i's that we denote by G_i. Up to permuting variables of $\mathbb{P}_k^m$ we can suppose that $G_0, \ldots, G_{m'}$ for some $0 \leq m' \leq m$ form a basis of $\operatorname{span}(G_0, \ldots, G_m)$ in $\bigoplus_{j=0}^N kY_j$. Using an automorphism of $\mathbb{P}_k^N$ we can suppose that $G_0, \ldots, G_{m'}$ are equal to $Y_0, \ldots, Y_{m'}$ and that $G_{m'+1}, \ldots, G_m$ are k-linear combination of the former, say $G_l = L_l(Y_0, \ldots, Y_{m'})$ for $m'+1 \leq l \leq m$. Say $k[X_0, \ldots, X_m] \xrightarrow{\varphi} k[Y_0, \ldots, Y_{m'}]$ is given sending $X_i \to Y_i$ if $i \leq m'$ and $X_i \to L_i(Y_0, \ldots, Y_{m'})$. Now, all in all the composition

$$k[X_0, \ldots, X_m] \xrightarrow{\text{permutation}} k[X_0, \ldots, X_m] \xrightarrow{\varphi} k[Y_0, \ldots, Y_{m'}]$$
$$\xrightarrow{\subseteq} k[Y_0, \ldots, Y_N] \xrightarrow{\text{automorphism}} k[Y_0, \ldots, Y_N] \to (k[X_0, \ldots, X_n])_d$$

induces by functoriality of Proj the map we started with.

(5) Explicitly decompose the map $\mathbb{P}_k^1 \to \mathbb{P}_k^2$ sending $[x : y]$ to $[x^2 : x^2 + y^2 + xy : 2x^2 + y^2 + xy]$, into the steps mentioned in the previous remark.