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AG II - Schemes and sheaves

Solutions – week 11

Exercise 1. *Tensor products,* Hom *and sheafification.*
Give examples of sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$ such that the tensor product presheaf and the presheaf

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U))$$

are not sheaves.
Hint: Play with $\mathcal{O}(-1)$ and $\mathcal{O}(1)$ on projective spaces. Recall the computation of global sections of those, exercise 5, week 10.

Solution key. Take for example $\mathcal{O}(1) \otimes \mathcal{O}(-1)$ and $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{O}(1))$. □

Exercise 2. *Effective Cartier divisors.* Let X be an integral scheme. A Cartier divisor on X represented by (f_i, U_i) is said to be *effective* if $f_i \in \mathcal{O}(U_i)$ for every i.

(1) Show, by looking at the ideal sheaf generated by the f_i's, that effective Cartier divisors are in one-to-one correspondence with ideal sheaves $\mathcal{I}$ that are a locally free sheaves of rank 1. We take this point of view in what follows.
(2) Let $\mathcal{L}$ be a locally free sheaf of rank 1. Show that $s\colon \mathcal{O}_X \to \mathcal{L}$ is non-zero if and only if the evaluation $\mathrm{ev}_s : \mathcal{L}^\vee \to \mathcal{O}_X$, defined by $\mathcal{L}^\vee(U) = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{L}_U, \mathcal{O}_u) \ni \varphi \mapsto \varphi(s)$ is injective.
(3) Fix a locally free sheaf $\mathcal{L}$ of rank 1. Deduce the following bijection,

$$\left(\Gamma(X, \mathcal{L}) \setminus \{0\}\right) / \mathcal{O}_X(X)^\times \to \{\text{Effective Cartier divisors } \mathcal{I} \text{ on } X \text{ with } \mathcal{I} \cong \mathcal{L}^\vee\}$$

that sends the class of a section s to $\mathrm{Im}(\mathrm{ev}_s)$.
(4) Suppose additionally that $\mathcal{O}_X(X)$ is a field. Show that if $\mathcal{L}$ is a locally free sheaf of rank 1 such that $\mathcal{L}$ and $\mathcal{L}^\vee$ have a non zero section, then $\mathcal{L} \cong \mathcal{O}_X$.
Hint: in this case both $\mathcal{L}$ and $\mathcal{L}^\vee$ correspond to effective Cartier divisors.
(5) Additionally assume that X is normal, Noetherian and integral. *Two Weil divisors are called linearly equivalent if their difference is the divisor of some rational function.* Let D be a Weil divisor on X. Show that map sending $f \in \Gamma(X, \mathcal{O}_X(D))$ to $\mathrm{div}(f) + D$ gives a one to one correspondence

$$\frac{\Gamma(X, \mathcal{O}_X(D)) \setminus \{0\}}{\mathcal{O}_X(X)^\times} \to \{\text{Effective Weil divisors linearly equivalent to } D\}.$$

Careful, hypothesis does not imply that $\mathcal{O}_X(D)$ defined as

$$\mathcal{O}_X(D)(U) = \{g \in K(X) \mid g \neq 0, (div(g) + D) \cap U \text{ is effective}\}$$

is a line bundle. So you have to prove it independently of item (3).

Solution key. (1) If (f_i, U_i) is an effective Cartier divisor, then we see that defining $f_i\mathcal{O}_{U_i} \subset \mathcal{O}_{U_i}$ defines a sub-ideal sheaf of $\mathcal{O}_X$ by gluing because f_i/f_j are units in functions on the intersection by assumption so $f_i\mathcal{O}_{U_{ij}} = f_j\mathcal{O}_{U_{ij}}$. Reciprocally given a locally free ideal sheaf $\mathcal{I}$, we know that there is an open cover (U_i) with $\mathcal{I}_{U_i} = f_i\mathcal{O}_{U_i}$. Now, (f_i, U_i) defines an effective Cartier divisor. Say we choose other generators (on a possible different open cover, but we deal with this case by taking a common refinement) $\mathcal{I}_{U_i} = f'_i\mathcal{O}_{U_i}$. Then there is $g_i \in \mathcal{O}_{U_i}(U_i)^\times$ with $f_i = g_i f'_i$. This implies that $(f_i, U_i) = (f'_i, U_i)$ has a Cartier divisor.

(2) This is a local check, so without loss of generality $\mathcal{L} = \mathcal{O}$ and $X = \operatorname{Spec}(A)$ is affine with A integral. So we are saying that $a \in A$ is non-zero if and only if

$$\begin{tikzcd} A \arrow[r, "1\mapsto \mathrm{id}"] \arrow[rr, bend right, "\cdot a"'] & \operatorname{Hom}_A(A, A) \arrow[r, "\mathrm{ev}_a"] & A \end{tikzcd}$$

the multiplication by a is injective.

(3) We produce an inverse. Say $\psi\colon \mathcal{I} \cong \mathcal{L}^\vee$. Then apply the $\operatorname{Hom}_{\mathcal{O}}(-, \mathcal{O})$ functor gives a map $s_\psi\colon \mathcal{O} \to \mathcal{I}^\vee \to \mathcal{L}$. Because an automorphism of a line bundle is always given by an element[1] in $\mathcal{O}_X(X)^\times$, it is clear that this inverse map does not depend on the choice of the isomorphism ψ.

(4) If both $\mathcal{L}$ and $\mathcal{L}^\vee$ have non-zero global sections, then say that $\mathcal{L} = \mathcal{I}$ an invertible ideal sheaf without loss of generality. But then $\mathcal{I}$ has a non-zero global section. Because we supposed that $\mathcal{O}_X(X)$ is a field, we see that 1 generated this ideal, concluding.

(5) About the inverse map, if D_1 is effective and linearly equivalent to D, this means that there exists $f \in K(X)$ such that $\operatorname{div}(f) + D = D_1$. So by definition f is a global section of $\mathcal{O}(D)$. Further details ommited. $\square$

Exercise 3. *Invertible sheaves and cocycles, a first encounter.* Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{L}$ be an invertible sheaf on X. Let (U_i) be a cover of X with trivializations $\varphi_i\colon \mathcal{L}_{U_i} \to \mathcal{O}_{U_i}$. We say that the associated cocycles $(\varphi \in \mathcal{O}_X(U_{ij})^\times)$ are defined to be $\varphi_i \circ \varphi_j^{-1}\colon \mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}}$ that we identify with $\varphi_{ij} \in \mathcal{O}_X(U_{ij})^\times$. Say $\mathcal{L}'$ is another invertible sheaf with associated cocycles (ψ_{ij}).

(1) Show that $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$ and $\varphi_{ii} = 1$.
(2) Show that the cocycles (φ_{ij}^{-1}) associated with $\mathcal{L}^\vee$ are , and the cocycles associated with $\mathcal{L} \otimes \mathcal{L}'$ are $(\varphi_{ij}\psi_{ij})$.
(3) Show that if for every i there is some $h_i \in \mathcal{O}(U_i)^\times$ such that $h_i\varphi_{ij}h_j^{-1} = \psi_{ij}$, then $\mathcal{L} \cong \mathcal{L}'$.

[1] An automorphism $\mathcal{L} \to \mathcal{L}$ is given locally by elements in $\mathcal{O}_{U_i}(U_i)^\times$ where $\mathcal{L}$ is trivial, but these will automatically glue. Indeed on the intersection they will be both equal to the restriction of the morphism.

We will go further in this study when introducing *first Čech cohomology.*

Solution key. (2) Denote suggestively $1/\varphi\colon \mathcal{O}_{U_i}$ the dual of φ^{-1}. This is a trivialization. Restricting to U_{ij} we see that the associated cocycle will be the dual of the multiplication by φ_{ij}^{-1} by functoriality of the dual. But the dual of the multiplication by an element identifies with the multiplication by this element. For the tensor product claim, note that the tensor product $\varphi_i \otimes \psi_j \colon \mathcal{L} \otimes \mathcal{L}' \to \mathcal{O}_X \otimes \mathcal{O}_X \xrightarrow{\text{mult}} \mathcal{O}_X$ is a trivialization.

(3) The condition ensures that the isomorphism $\psi_i^{-1} h_i \varphi_i \colon \mathcal{L}_{U_i} \to \mathcal{L}'_{U_i}$ glues.

□

Exercise 4. *Extension of coherent sheaves.* The goal is to show that if X is a Noetherian scheme, U an open subset and $\mathcal{F}$ is a coherent sheaf on U, then there is a coherent sheaf $\mathcal{G}$ on X such that $\mathcal{G}_{|U} \cong \mathcal{F}$.

(1) Show that on a Noetherian scheme X and $\mathcal{F}$ coherent sheaf, then if
$$\sum_i \mathcal{F}_i = \mathcal{F}$$
where $(\mathcal{F}_i)_{i \in I}$ are sub-coherent sheaves, then there exist a finite refinement $J \subset I$ such that $\sum_{j \in J} \mathcal{F}_j = \mathcal{F}$.

(2) Show that on a Noetherian affine scheme, every quasi-coherent sheaf is the direct colimit of it's coherent sub-sheaves. *Hint: Use the equivalence of categories with modules on global sections.*

(3) Let X be affine and $\iota\colon U \to X$ be an open subcheme. Show the claim in this case. *Hint: Show that $\iota_* \mathcal{F}$ is quasi-coherent, and then use a combination of (1) and (2) to conclude.*

(4) Show the claim in the general case of the statement of the exercise by induction on the number of open affines that are required to cover X. (Being covered by one open affine being the base case of the induction, and is the previous point. The rest is an induction play, see Hint.) *Hint: Say $X = X_1 \cup X_2$ where X_1 and X_2 are open subschemes that can be covered by strictly less open affines than X. By induction extend $\mathcal{F}_{X_1 \cap U}$ to a coherent sheaf $\mathcal{G}_1$ defined on X_1. By gluing $\mathcal{F}$ and $\mathcal{G}_1$ it defines a coherent sheaf $\mathcal{G}'$ defined on $X_1 \cup U$. Now, extend $\mathcal{G}'_{|X_2 \cap (X_1 \cup U)}$ to a coherent sheaf $\mathcal{G}_2$ on X_2. Conclude by gluing $\mathcal{G}_1$ and $\mathcal{G}_2$ to a coherent sheaf on X.*

As an application, show that any quasi-coherent sheaf on a Noetherian scheme is a direct colimit of sub-coherent sheaves.

Solution key. (1) Let $\mathcal{F}$ be a coherent sheaf on a Noetherian scheme X. Suppose that
$$\sum_i \mathcal{F}_i = \mathcal{F}$$
where $(\mathcal{F}_i)_{i \in I}$ are sub-coherent sheaves. Then there exist a finite refinement $J \subset I$ such that $\sum_{j \in J} \mathcal{F}_j = \mathcal{F}$. Indeed, as we can cover X by *finitely* many open affines, we may prove that we can find

a finite refinement for each open affine. But now this follows just from the fact that a coherent sheaf on a Noetherian affine scheme amounts to a finite module M. Each generator of M is a finite sum of sections of the $\mathcal{F}_i$'s.

(2) Recall that a Noetherian affine scheme $\mathrm{Spec}(A)$, quasi-coherent sheaves are equivalent to A-modules and coherent sheaves are equivalent to finite A-modules. Therefore to check that

$$\bigcup \mathcal{F}_\alpha = \mathcal{F}$$

it suffices to check it on global sections. But it amounts to say that an A-module is the union of it's sub finite A-modules.

(3) Cover U by *finitely* many open affine schemes $(U_i)_i$. Note that because an affine scheme is separated, intersections U_{ij} are also affine. We may use that affine morphism preserves quasi-coherence, and that quasi-coherent sheaves are stable by kernels. Denote by $\iota_i \colon U_i \to X$ and $\iota_{ij} \colon U_{ij} \to X$ inclusions. Now remark that

$$\iota_*\mathcal{F} = \ker\left(\bigoplus_i \iota_{i*}\mathcal{F} \to \bigoplus_{ij} \iota_{ij,*}\mathcal{F}\right)$$

where the map sends $(f_i) \mapsto (f_i - f_j)$.

We therefore know that this sheaf is the union of it's coherent subsheaves by (a)

$$\bigcup \mathcal{F}'_\alpha = \iota_*\mathcal{F}.$$

Because $\mathcal{F}_U$ is coherent, by the compacity remark, there exists finitely many $\alpha_1, \ldots, \alpha_n$ such that $\mathcal{F}_U$ is the sum of the $\mathcal{F}'_{\alpha_i,U}$. If we set $\mathcal{F}'$ to be the sum of these we get the claim.

(4) $\mathcal{G}$ is the union of it's coherent sub-modules, say $\mathcal{G}_\alpha$. By the compacity remark above, we may sum finitely many such that $\sum_i \mathcal{G}_{\alpha_i,U} = \mathcal{F}$.

(d) We proceed by induction on the number of affine schemes that can cover X. If X is affine, we are done. Otherwise, we can write

$$X = X_1 \cup X_2$$

with X_1 and X_2 open subschemes that can be written as union of stricly less open affines. Find a coherent sheaf on X_1 that we denote by $\mathcal{F}_1 \subset \mathcal{G}_{X_1}$ that extends $\mathcal{F}_{|X_1 \cap U}$ from $X_1 \cap U$ to X_1 by induction. Note that this defines a coherent sheaf on $X_1 \cup U$ by gluing $\mathcal{F}_1$ and $\mathcal{F}$. Denote this sheaf by $\mathcal{F}'_1$. Now extend $\mathcal{F}'_{1,|X_2 \cap (U \cup X_1)}$ from $X_2 \cap (U \cup X_1)$ to X_2 by induction to a coherent sheaf $\mathcal{F}_2$. Now remark that $\mathcal{F}_2$ and $\mathcal{F}'_1$ glue to the desired sheaf $\mathcal{F}'$.

(5) Let $s \in \mathcal{F}(U)$. Consider the coherent sheaf generated by s on U. Extend it to a coherent sheaf on X.

□

Exercise 5. *Divisors on regular curves.* Let k be an algebraically closed field. We say that C is a *regular k-curve over k* is a one dimensional separated, integral and regular scheme over k. Weil (=Cartier in this case) divisors are then of the form

$$D = \sum_i n_i x_i$$

for x_i being closed points of C. We define the *degree* of a divisor $D = \sum_i n_i x_i$ to be

$$\deg(D) = \sum_i n_i \in \mathbb{Z}.$$

Let $f\colon C' \to C$ a finite k-morphism between regular k-curves. We define the *pullback* of an irreducible divisor (=closed point)

$$f^* x = \sum_{y \in C'_{cl} \text{ s.t. } x = f(y)} v_y(f^\sharp(t_x)) y.$$

where $f^\sharp$ denotes the induced map at the local ring. Here, t_x denotes a generator of $\mathfrak{m}_x$ – this well defined because the choice of a generator is up to a unit. We extend f^* by linearity to $\operatorname{div}(C)$.

(1) Show that the pullback of a principal divisor is principal, implying that f^* factors through

$$f^*\colon \operatorname{Cl}(C) \to \operatorname{Cl}(C').$$

(2) Show that if the degree of the map ($= [K(C')\colon K(C)]$) is d, then $\deg(f^* D) = d \deg(D)$. *Hint: it suffices to show the claim for $D = x$ a closed point by linearity.*

(3) Assume now that C is also proper. Using the equivalence of categories seen in lecture on curves (that you can assume) between k-fields of k-transcendence degree 1 and regular proper k-curves, show that for every $t \in K(C)\backslash k$ we have a map $f_t\colon C \to \mathbb{P}^1_k$ from the inclusion $k(t) \subset K(C)$ such that $f^*(0 - \infty) = (f)$ where 0 denotes $V(f)$ in $\operatorname{Spec}(k[f]) \subset \mathbb{P}^1_k$ and ∞ denotes $V(1/f)$ in $\operatorname{Spec}(k[1/f]) \subset \mathbb{P}^1_k$. deduce that $\deg((f)) = 0$, and that therefore deg factor through

$$\deg\colon \operatorname{Cl}(C) \to \mathbb{Z}.$$

Solution key. (1) One sees that the pullback of $\operatorname{div}(g)$ for some $g \in K(C)$ is given by $\operatorname{div}(f^\sharp(g))$ where here $f^\sharp$ denotes the induced map at field of fractions.

(2) The fiber at x is a finite k-algebra of dimension d. Because k is algebraically closed, such algebras are isomorphic to

$$\prod k[t]/(t_i)^n_i$$

which has much factors has the set theoretic cardinality of the fiber and necessarily $\sum n_i = d$.

(3) If $t \in K(C) \setminus k$, because k is algebraically closed, t is transcendent. So $k(t) \to K(C)$ induces a map $C \to \mathbb{P}^1_k$ by the equivalence of categories. Now, it amounts to noticing that $\operatorname{div}(t) = 0 - \infty$ and using the preceding point.

□

Exercise 6. *Segre viewed with line bundles.* Fix an algebraically closed field k. Denote the projection of $\mathbb{P}^1 \times_k \mathbb{P}^1$ to the first and second factor by p_1 and p_2 respectively. View the first and second copy of $\mathbb{P}^1$ in the product as $\mathrm{Proj}(k[x_0, x_1])$ and $\mathrm{Proj}(k[y_0, y_1])$ respectively. Show that the global sections $p_1^*(x_i) \otimes p_2^*(y_j)$ for $0 \leq i, j \leq 1$ of $p_1^*(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$ give a closed embedding of $\mathbb{P}^1 \times_k \mathbb{P}^1$ in $\mathbb{P}^3$.

Solution key. Affine locally this is a Segre embedding. $\square$