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AG II - Schemes and sheaves

Solutions– week 10

Exercise 1. *Functoriality of $\mathcal{O}(n)$.* Let R and S be $\mathbb{N}$-graded rings, where R is generated in degree 1, so that $\mathcal{O}(n)$ is a line bundle for each $n \in \mathbb{Z}$. Let $f: R \to S$ be an homogeneous map of degree $d \geq 1$, see Exercise 5, week 4. Denote by $g: U \to \operatorname{Proj}(R)$ the induced map at Proj from functoriality of Proj. Show then that for $n \geq 0$ we have $g^*\mathcal{O}(n) = \mathcal{O}(nd)_{|U}$.
Hint: check the claim on cocycles.
Solution key.
We first tacke the case where the the map is a graded ring map (homogeneous of degree 1).
As R is genererated in degree 1, $\mathcal{O}_{\operatorname{Proj}(R)}(1)$ is a line bundle. Therefore it's pullback to U is also a line bundle. Note that on U, because U is covered by D_+ of degree 1 elements that $\mathcal{O}_{\operatorname{Proj}(S)}(1)_{|U}$ is a line bundle. To check that they are isomorphic, we can compare cocycles. Denote by $(R_1)_h$ homogeneous elements of degree 1. Because

$$\bigcup_{r \in (R_1)_h} D_+(r) = \operatorname{Proj}(R)$$

then $U = \bigcup_{r \in (R_1)_h} D_+(r)$ by definition. Write $U_r = D_+(r)$. Then cocycles of both invertible sheaves are $\varphi_{rr'} = r/r'$.
Another way of seeing this is by what follows. Let $r \in (R_1)_h$. Define the multiplcation map

$$S_{(r)} \otimes R(n)_{(r)} \to S(n)_{(r)}$$

which glues to a map $g^*\mathcal{O}_{\operatorname{Proj}(R)}(1)_{|U} \to \mathcal{O}_{\operatorname{Proj}(S)}(1)_{|U}$ is seen to be a bijection sending an element s/r^d (with s homegeneous of degree $d+n$) to $s/r^{d+n} \otimes r^{d+n}/r^d$.
Now we tackle the general case of an homegeneous map of degree d. It suffices to address now the case of the isomorphism $\operatorname{Proj}(R) \to \operatorname{Proj}(R_d)$ by the inclusion of $v_d: R_d \to R$ the d-th Veronese subring. The claim follows from the fact that a degree n element in R_d is of degree nd in R. Indeed this leads to $(R_d(n))_{(r)} = R(nd)_{(r)}$.

Exercise 2. *A principal divisor is effective where it has no poles.* Let X be a Noetherian, normal and integral scheme. Recall that for normal Noetherian domain A, the ring A is the intersection of $A_\mathfrak{p}$ where $\operatorname{ht}(\mathfrak{p}) = 1$.
Let $f \in K(X)$. Let $U \subset X$ open. Show that if $\operatorname{div}(f)_{|U} \geq 0$ then $f \in \mathcal{O}_X(U)$. If $\operatorname{div}(f)_{|U} = 0$ then $f \in \mathcal{O}(U)^\times$.
Solution Key.
This is a local statement. So we can suppose that $U = \operatorname{Spec}(A)$ is affine. But then $A = \cap_\mathfrak{p} A_\mathfrak{p}$ where the intersection is taken on height 1 primes concludes.

Exercise 3. *Divisors that are not Cartier.* Let k be a field and $X = V(xy - zw)$ in $\mathbb{A}^4_k$. Note that X is integral and regular in codimension 1.

(1) Show that the closed subsets in X defined by $x = z = 0$ and $x = w = 0$ are prime divisors that are not Cartier. Denote by D_z and D_w these divisors.
(2) Show that $D_z + D_w$ is a Cartier divisor.

Solution key.
Note that D_z and D_w are prime Weil divisors isomorphic to $\mathbb{A}^2_k$ because

$$k[x, y, z, w]/(xy - zw, x, z) = k[y, w] \quad [x, y, z, w]/(xy - zw, x, w) = k[y, z].$$

If $I_z = (x, z)$ or $I_w = (x, w)$ where to define Cartier divisors, then these ideals would be locally principal, in the sense that in sufficiently small affine open sets, these ideals would be principal in the ring of functions of these opens. In particular, in each local ring these ideals would be principal. But at the local ring at the origin $\mathfrak{m}$, note that $\mathfrak{m}/\mathfrak{m}^2$ is a k-vector space of dimension 4, and the basis is given by the images of x, y, z, w. If I_z or I_w where to be generated by one element in this local ring, then the k-vector space spanned by the images of x, z and x, y respectively in $\mathfrak{m}/\mathfrak{m}^2$ would be of k-dimension 1, a contradiction.
On the other hand, note that $V(x) = D_z + D_w$, implying that this divisor is Cartier.

Exercise 4. *Exact sequence for class groups.* Let X be an integral separated scheme which is regular in codimension 1. Let Z be a proper closed subset of X and $U = X \setminus Z$.

(1) Show that $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ defined by $\sum n_i D_i \mapsto \sum n_i(D_i \cap U)$ is surjective.
(2) If $\mathrm{codim}(Z, X) \leq 2$, show that that this map is also injective.
(3) If $\mathrm{codim}(Z, X) = 1$ and Z is irreducible, show that there is an exact sequence
$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 1$$
where $\mathbb{Z} \to \mathrm{Cl}(X)$ send 1 to Z.
(4) Let k be a field. Let Z be the zero set of an irreducible homogeneous polynomial of degree d in $\mathbb{P}^n_k$. Deduce that $\mathrm{Cl}(\mathbb{P}^n_k \setminus Z) \cong \mathbb{Z}/d\mathbb{Z}$.

Hint: You may look at chapter II.6 of Hartshorne.
Solution key.
The last statement follows frome example seeing that that $V_+(F)$ for F irreducible homegenous of degree d correspond to $\mathcal{O}(d)$ via the identification of Picard groups and Cartier class groups, and the $\mathcal{O}(d)$ is d times the generator of the Picard group which is infinite cyclic.

Exercise 5. Let A be a ring and $R_\bullet$ the graded ring $A[x_0, \ldots, x_n]$ with $\deg(x_i) = 1$. Show that the natural map

$$R_m \to \Gamma(\mathrm{Proj}(R), \mathcal{O}(m))$$

is an isomorphism for $m \in \mathbb{Z}$. *Hint: Use the usual cover and the sheaf property.*

Solution key.
Consider the cover of Proj(A) by $D_+(x_i)$ for $i = 0, \dots, n$. Recall that

$$\mathcal{O}(m)_{|D_+(x_i)}(D_+(x_i)) = x_i^m R_{(x_i)}.$$

First note that the natural map is given by sending $f \in R_m$ to global section defined by $f \in x_i^m R_{(x_i)}$.
By the sheaf property, a global section of $\mathcal{O}(m)$ corresponds to a collection $f_i/x_i^{n_i}$ where $\deg(f_i) = m+n_i$ that agrees on intersections. Because we work in a polynomial ring, we can suppose that x_i does not divide f_i. Agreeing on intersections says that

$$f_i x_j^{n_j} = f_j x_i^{n_i}.$$

Because x_i, x_j is a regular sequence we deduce that $n_j = n_i = 0$. Therefore we deduce that $f_i = f_j = f$ an homogeneous element of degree m, concluding.

Exercise 6. *Support of coherent sheaves.* Let X be a locally Noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. We define

$$\operatorname{supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}$$

(1) Let A be a ring and M a finitely generated module. Show that $\operatorname{supp}(M)$ is closed.
(2) In the same setup as in item (1), show that $\operatorname{supp}(M) = V(\operatorname{Ann}(M))$, where

$$\operatorname{Ann}(M) = \{f \in A \mid fM = 0\}.$$

(3) Let A be a Noetherian ring, $f \in A$ and M be a finitely generated module. Show $\operatorname{Ann}(M)_f = \operatorname{Ann}(M_f)$.
(4) Let X be a locally Noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. Using the preceding point, define a quasi-coherent sheaf of ideals $\operatorname{Ann}(\mathcal{F})$. Show that $V(\operatorname{Ann}(\mathcal{F})) = \operatorname{supp}(\mathcal{F})$.

Solution key.

(1) Let $m_1, \dots, m_r$ be generators of M. Note that the complement of $\operatorname{supp}(M)$ is the locus of $\mathfrak{p}$'s where $(m_i)_\mathfrak{p} = 0$ for all i. But if this holds this means that there is some $a \in A \setminus \mathfrak{p}$ such that $am_i = 0$. Then m_i is zero on $D_+(a)$, showing that the complement is open being the finite intersection of the loci where m_i's vanish.
(2) We work on complements. We want to show that the complement of $\operatorname{supp}(M)$ is $\bigcup_{f \in \operatorname{Ann}(M)} D(f)$. Note that

$$\bigcup_{f \in \operatorname{Ann}(M)} D(f) \subset \operatorname{Spec}(A) \setminus \operatorname{supp}(M),$$

because if $M_f = 0$, then any further localization is zero. For the other inclusion, if $M_\mathfrak{p} = 0$, then if $f_i \in A \setminus \mathfrak{p}$ is an element killing m_i, then the product f of the f_i's is an element not in $\mathfrak{p}$ with $fM = 0$.
(3) Note that $\operatorname{Ann}(M)_f \subset \operatorname{Ann}(M_f)$. If $(g/f^r)M_f = 0$, then $(g/f^r)m_i = 0$, implying that $gf^{n_i}m_i = 0$ for some n_i. Taking a big enough power shows the surjectivity of the map.
(4) Immediate from last observation.

Remark. In this case, we then call $V(\mathrm{Ann}(\mathcal{F}))$ with it's natural scheme structure coming from the quasi-coherent sheaf of ideals $\mathrm{Ann}(\mathcal{F})$ the scheme theoretic support of $\mathcal{F}$.

Exercise 7. *Torsion free sheaves.* Let X be an integral scheme with generic point η. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is torsion free if $\mathcal{F}(U)$ is a torsion free $\mathcal{O}(U)$-module for all opens $U \subset X$.

(1) Let $\mathcal{F}$ be any quasi-coherent sheaf. Show that $\mathcal{F}_{tors} \subset \mathcal{F}$, where $s \in \mathcal{F}_{tors}(U)$ if $s \mapsto 0$ along $\mathcal{F} \to \mathcal{F}_\eta$, is a quasi-coherent sheaf and that $\mathcal{F}/\mathcal{F}_{tors}$ is torsion free.
(2) Show that a map between torsion free sheaves is injective if and only if it is injective at a stalk at some point $x \in X$.
(3) Deduce that a map between locally free sheaves of rank 1 is injective or zero.

Solution key.

(1) It's defined to be the kernel of $\mathcal{F} \to \iota_\nu \mathcal{O}_{\mathrm{Spec}(K(X))}$ – the map ι_ν denotes $\mathrm{Spec}(K(X)) \to X$. It is therefore coherent as the kernel of a map between quasi-coherent sheaves.

    Note that being torsion free is a property that can be checked at stalks. This implies the second part of the statement.
(2) Follows from the fact that in the case the stalk map $\mathcal{F}(U) \to \mathcal{F}_\nu$ is injective.
(3) If the map is not injective, it will not be injective at the generic point but at the generic point we have a self map of $K(x)$-modules of $K(X)$ and the only way that this map is not injective is that this is the zero map. This concludes.

Exercise 8. *Generic flatness.* Let X be a reduced Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X.

(1) Show that there is an non empty open U such that $\mathcal{F}$ is locally free (possibly zero).

    Use Exercise 4.(3), week 9.
(2) Show by Noetherian induction[1] on X that there is a finite partition of X by locally closed subschemes (X_i) with the reduced scheme structure such that $\mathcal{F}$ is locally free when restricted (meaning taking the pullback) to X_i.

Solution Key.
The function $\varphi(x) = \dim_{k(x)}(\mathcal{F} \otimes_{\mathcal{O}_X} k(x))$ takes a minimal value in $\mathbb{N}$ (could be zero). Because φ is semi-continuous, the locus where φ is equal to this minimal value is open. By the last part of the exercise on this topic, we get the claim. (Remark that there is no worry about connectedness.)
The previous claim shows the basis and the induction of Noetherian induction.

[1] see *Hartshorne, II.3.16*