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AG II - Schemes and sheaves

Exercise to hand in. *Morphisms between projective spaces, again.*
Let k be a field. You may find part (1), (2) of the exercise useful while solving part (3) of the exercise.

(1) (**Graded Nakayama**) Let $R = \underset{n\in\mathbb{N}}{\oplus} R_n$ be a graded ring and $M = \underset{n\in\mathbb{N}}{\oplus} M_n$ be a graded R module, $J \subseteq R_+$ be a homogeneous ideal of R. If $M = JM$, then show that $M = 0$. Recall that R_+ is the homogeneous ideal $\underset{n>0}{\oplus} R_n$ of R.

(2) Let R be a Noetherian ring generated in degree 1 as an R_0-algebra.[1] Let $F_0, \dots, F_m$ be homogeneous elements of strictly positive degrees of R. Assume that the radical of $(F_0, \dots, F_r)$ is R_+. Show that the graded ring inclusion $R_0[F_0, \dots, F_r] \to R$ is a finite ring map. You may want to use part (1).

(3) Given a morphism of k-schemes $f\colon \mathbb{P}^n_k \to \mathbb{P}^m_k$, show that the image is either a point; and if the image is not a point, then $m \geq n$ and the image has dimension n. In the second case, show that the morphism is finite. *Hint: we have $f^*\mathcal{O}_{\mathbb{P}^m_k}(1) \cong \mathcal{O}_{\mathbb{P}^n_k}(d)$ for some $d \geq 0$. Break the study into two cases: $d = 0$ and $d \geq 1$. In this last case show that the polynomials $F_0, \dots, F_m$ homogeneous of degree d which defines the map generates an ideal which radical is $k[X_0, \dots, X_n]_+$.*

(4) Identify $\mathbb{P}^n_k = \operatorname{Proj}(k[x_0, \dots, x_n])$. Show that the projection map from $\mathbb{P}^n_k - [0 : 0 : \dots : 1]$ to $\mathbb{P}^{n-1}_k$ given by the sections $x_0, x_1, \dots x_{n-1} \in \mathcal{O}_{\mathbb{P}^n_k}(1)$ cannot be extended to $\mathbb{P}^n_k$.

Remark. When the image of a map $\mathbb{P}^n_k \to \mathbb{P}^m_k$ is not a point, the map $\mathbb{P}^n_k \to \mathbb{P}^m_k$ can be written as a composition of (1) a Veronese embedding $\mathbb{P}^n_k \to \mathbb{P}^N_k$, for some N, (2) an automorphism of $\mathbb{P}^N_k$, (3) a projection map

$$\mathbb{P}^N_k - V(X_0, \dots, X_{m'}) \to \mathbb{P}^{m'}_k,$$

sending $(x_0 : \dots : x_N)$ to $(x_0 : \dots : x_{m'})$, for some $m' \leq m$, (4) a linear embedding $\mathbb{P}^{m'}_k \to \mathbb{P}^m_k$ – here, a linear embedding is map that sends $\underline{x} = (x_0 : \dots : x_{m'})$ to $(x_0 : \dots : x_{m'} : L_{m'+1}(\underline{x}) : \dots : L_m(\underline{x}))$, where the L_j's are some linear polynomials in $m' + 1$ variable with k-coefficients – and (5) an automorphism which is a permutation of variables $\mathbb{P}^m_k$.
Indeed, a map $\mathbb{P}^n_k \to \mathbb{P}^m_k$ is given by homogeneous $m + 1$ polynomials of some fixed degree d, that we denote by $F_0, \dots, F_m$. By functoriality of Proj we can express this map (where the first map is given by $X_i \mapsto F_i$)

$$k[X_0, \dots, X_m] \to (k[X_0, \dots, X_n])_d \subset k[X_0, \dots, X_n].$$

So we can factor by a Veronese embedding

$$k[X_0, \dots, X_m] \to k[Y_j]_{j=0}^N \to (k[X_0, \dots, X_n])_d$$

[1] It is enough to assume that R is $\mathbb{N}$-graded Noetherian.

sending X_i to lifts of F_i's that we denote by G_i. Up to permuting variables of $\mathbb{P}^m_k$ we can suppose that $G_0, \dots, G_{m'}$ for some $0 \le m' \le m$ form a basis of $\operatorname{span}(G_0, \dots, G_m)$ in $\bigoplus_{j=0}^N kY_j$. Using an automorphism of $\mathbb{P}^N_k$ we can suppose that $G_0, \dots, G_{m'}$ are equal to $Y_0, \dots, Y_{m'}$ and that $G_{m'+1}, \dots, G_m$ are k-linear combination of the former, say $G_l = L_l(Y_0, \dots, Y_{m'})$ for $m'+1 \le l \le m$. Say $k[X_0, \dots, X_m] \xrightarrow{\varphi} k[Y_0, \dots, Y_{m'}]$ is given sending $X_i \to Y_i$ if $i \le m'$ and $X_i \to L_i(Y_0, \dots, Y_{m'})$. Now, all in all the composition

$$\begin{aligned} &k[X_0, \dots, X_m] \xrightarrow{\text{permutation}} k[X_0, \dots, X_m] \xrightarrow{\varphi} k[Y_0, \dots, Y_{m'}] \\ &\xrightarrow{\subseteq} k[Y_0, \dots, Y_N] \xrightarrow{\text{automorphism}} k[Y_0, \dots, Y_N] \to (k[X_0, \dots, X_n])_d \end{aligned}$$

induces by functoriality of Proj the map we started with.

(5) Explicitly decompose the map $\mathbb{P}^1_k \to \mathbb{P}^2_k$ sending $[x : y]$ to $[x^2 : x^2+y^2+xy : 2x^2+y^2+xy]$, into the steps mentioned in the previous remark.

Solution key. (1) By contradiction if M is not zero, we see that the minimal degree of non-zero elements of M and JM are different.

(2) Because of the Noetherian and the generated in degree 1 hypothesis there is some N with

$$(R_+)^N = \bigoplus_{d \ge N} R_d \subset (F_0, \dots, F_r).$$

Note that also that $\bigoplus_{i=0}^{N-1} R_i$ is an R_0 finite module. So as $R_0[F_0, \dots, F_r]$ modules we have

$$R = \bigoplus_{i=0}^{N-1} R_i + (F_0, \dots, F_r)$$

concluding using the above version of Nakayama.

(3) If $d = 0$, then the map is given by $\lambda_0, \dots, \lambda_m \in k$, and using the construction seen in class, we see that we have a factorization

$$\begin{array}{ccc} \mathbb{P}^n_k & \longrightarrow & \mathbb{P}^m_k \\ \downarrow & \nearrow_{[\lambda_0, \dots, \lambda_m]} & \\ \operatorname{Spec}(k) & & \end{array}$$

If $d \ge 1$, we see that the map is induced by functoriality of Proj by

$$k[X_0, \dots, X_m] \to k[X_0, \dots, X_n]$$

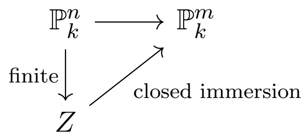
sending $X_i \mapsto F_i$. We can factorize

$$\begin{array}{ccc} k[X_0, \dots, X_m] & \longrightarrow & k[X_0, \dots, X_n] \\ {\scriptstyle\text{surjective}}\downarrow & \nearrow_{\text{inclusion}} & \\ k[F_0, \dots, F_m] & & \end{array}$$

Note that because the map is globally defined on $\mathbb{P}^n_k$, $V_+(F_0, \dots, F_m) = \emptyset$, implying that the radical of $(F_0, \dots, F_m)$ is an homogeneous ideal

$I \subset (X_0, \ldots, X_n)$ with the property that $I_{(X_i)} = k[\frac{X_j}{X_i}]$ for every i. In particular there is some $g \in I$ with $g/X_i^m = 1$, for some $m \geq 1$ because elements of I are of strictly positive degree, implying that $X_i^m \in I$. As the ideal is radical, we see that $X_i \in I$. Because i was arbitrary in this reasoning we get $(X_0, \ldots, X_n) \subset I$, getting $(X_0, \ldots, X_n) = I$. This shows using the preceding point that $k[F_0, \ldots, F_m] \subset k[X_0, \ldots, X_n]$ is finite.

It implies that we have a factorization of the map



concluding.

(4) If so, as the map would not be constant, then the image would be of dimension n, a contradiction.

(5) No permutation is needed as the sum of x^2 and $x^2 + y^2 + xy$ is the last one. Say t_0, t_1, t_2 are the coordinates of the polynominal ring surjecting to $k[x^2, y^2, xy]$ in the suggested construction. Let $\varphi \colon k[x, y, z]$ with $x \mapsto t_0$ and $y \mapsto t_1$ and $z \mapsto t_0 + t_1$. Now compose with the automorphism

$$t_0 \mapsto t_0 \quad t_1 \mapsto t_0 + t_1 + t_2 \quad t_2 \mapsto t_2$$

to conclude.

$\square$