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AG II - Schemes and sheaves

Exercise to hand in. *Morphisms and maps between projective spaces.*
Let k be a field, and $m < n$ two positive integers. Consider the two k-algebra morphisms given by the natural inclusion $\phi\colon k[x_0, \dots, x_m] \hookrightarrow k_[x_0, \dots, x_n]$ and the natural quotient $\psi\colon k[x_0, \dots, x_n] \twoheadrightarrow k_[x_0, \dots, x_m]$. Let $f\colon \mathbb{A}^{n+1}_k \to \mathbb{A}^{m+1}_k$ and $g\colon \mathbb{A}^{m+1}_k \to \mathbb{A}^{n+1}_k$ denote the corresponding morphisms of affine spaces, and let $\pi\colon \mathbb{P}^n_k \dashrightarrow \mathbb{P}^m_k$ and $\iota\colon \mathbb{P}^m_k \dashrightarrow \mathbb{P}^n_k$ be the corresponding rational maps between projective spaces, obtained by functoriality of Proj, see Exercise 5, week 5.

(1) Assuming that k is algebraically closed so that we can represent closed points with Cartesian coordinates $(a_0, \dots, a_m)$ and $(b_0, \dots, b_n)$, describe the morphisms f and g at the level of coordinates.
(2) Show that ι is a morphism and a closed embedding, and show that π is not everywhere defined. Furthermore, show that the locus where π is not defined is a copy of $\mathbb{P}^{n-m-1}_k$. Lastly, assuming that k is algebraically closed so that we can represent closed points with projective Cartesian coordinates $[a_0 : \ldots : a_m]$ and $[b_0 : \ldots : b_n]$, describe the two maps at the level of coordinates.

In general, π is called *projection from an* $(n-m-1)$*-plane.* The fibers over closed points of this rational map (i.e., the closure of the fibers of the morphism defined on the domain of π) are copies of $\mathbb{P}^{n-m-1}_k$. For instance, if $n = 2$ and $m = 1$, it is a projection from a the point $[0 : 0 : 1]$.
From the point of view of linear systems (cf. Ch. II.7 in Hartshorne), the rational map π is defined by a proper subspace of $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$, namely by those global sections that vanish along the linear subspace we are projecting from. For instance, in the case $n = 2$ and $m = 1$, the rational map π is defined by considering the sections of $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$ corresponding (cf. Exercise 2) to the lines through the point $[0 : 0 : 1]$.
In the following, $\mathbb{P}^{n-m-1}_k$ will denote the copy of the projective $(n-m-1)$-space along which π is not defined.

(3) Show that $\iota^*\mathcal{O}_{\mathbb{P}^n_k}(1) = \mathcal{O}_{\mathbb{P}^m_k}(1)$. *Hint: you can use Exercise 1.*
(4) Show that $\mathcal{O}_{\mathbb{P}^n}(1)|_{\mathbb{P}^n_k \setminus \mathbb{P}^{n-m-1}_k}$ is isomorphic to $\pi^*\mathcal{O}_{\mathbb{P}^m_k}(1)$. *Hint: you can use Exercise 1.*

In the following, we focus on the case $n = 2$ and $m = 1$, and we further assume that k is algebraically closed. We will denote by $P = [0 : 0 : 1]$ the copy of $\mathbb{P}^{2-1-1}_k$ (i.e., a point) along which π is not defined. We let C_1 be the conic with equation $x_2^2 - x_0x_1 = 0$, which corresponds to the Veronese embedding of $\mathbb{P}^1_k$ in $\mathbb{P}^2_k$ (cf. Exercise 6 in sheet 4)[1]. Then, we denote by C_2 the conic with equation $x_0^2 - x_1x_2 = 0$. Notice that $P \in C_2$ and $P \notin C_1$.

(5) Show that $\mathcal{O}_{\mathbb{P}^2_k}(1)|_{C_1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1_k}(2)$, where we identify $\mathbb{P}^1_k$ with C_1 via the Veronese embedding. *Hint: you can use Exercise 1.*

[1]More precisely we ware talking about the one induced by Proj by $x_0 \mapsto x_0^2$, $x_1 \mapsto x_1^2$ and $x_2 \mapsto x_0x_1$.

(6) Show that $\pi|_{C_1} : C_1 \to \mathbb{P}^1_k$ is finite of degree 2. *Hint: via isomorphism given by the Veronese embedding, you can identify* π_{C_1} *with one of the morphisms in Exercise 1 in sheet 7.*

(7) Show that $\pi|_{C_2} : C_2 \setminus \{P\} \to \mathbb{P}^1_k$ extends uniquely to an isomorphism $\pi|_{C_2} : C_2 \to \mathbb{P}^1_k$. *Hint: Define a map* $D_+(x_2) \cap C_2 \to D_+(x_0) \subset \mathbb{P}^1_k$ *that glues with* $\pi|_{C_2} : C_2 \setminus \{P\} \to \mathbb{P}^1_k$. *Note that you are forced to send* $\frac{x_1}{x_0} \in K(\mathbb{P}^1_k)$ *to* $\frac{x_1}{x_0} = \frac{x_0}{x_2} \in K(C_2)$ *which ensures unicity.*

(8) Show that $\mathcal{O}_{\mathbb{P}^2_k}(1)|_{C_2}$ is not isomorphic to $(\pi|_{C_2})^*\mathcal{O}_{\mathbb{P}^1_k}(1)$. *Hint: you can use Exercise 1.*

The morphism $\pi|_{C_2}$ is nothing but the *stereographic projection*. Indeed, the fibers of π (i.e., the closure of the fibers of the morphism $\mathbb{P}^2_k \setminus \{P\} \to \mathbb{P}^1_k$) are lines. In the case of C_1, these lines intersect C_1 in 2 (by Bézout's theorem) distinct points, and these points vary as we vary the target point in $\mathbb{P}^1_k$. On the other hand, in the case of C_2, one of the two points is always P. Thus, we get a morphism from $C_2 \setminus \{P\}$ which is an isomorphism with its image, which in turn extends to the whole C_2 (ancient Greeks just settled for a bijection...).
More generally, if we have a regular conic C with a k-rational point P, the projection from P always induces and isomorphism with $\mathbb{P}^1_k$.

Solution key. (1) *(Mathis)*

Since coordinates $(a_0, ..., a_m)$ simply correspond to the maximal ideal $(x_0 - a_0, ..., x_m - a_m)$ (and similarly for $(b_0, ..., b_n)$), it is easy to see that

$$f(b_0, ..., b_n) = (b_0, ..., b_m), \quad g(a_0, ..., a_m) = (a_0, ..., a_m, 0, ..., 0)$$

(2) *(Mathis)*

By Exercise 5 of sheet 4, since ι is induced by the quotient map ψ, which is surjective on each graded piece, ι is a morphism and a topological closed embedding. On the other hand π is not everywhere defined: the point with coordinates $[0 : ... : 0 : 1]$ would be sent via π to $[0 : ... : 0]$. To be more general, letting $S = k[x_0, ..., x_m]^+$ be the irrelevant ideal, $\varphi(A_+) \subset \mathfrak{p}$ where $\mathfrak{p}$ is any homogeneous prime generated by elements of $k[x_{m+1}, ..., x_n]$. Thus π is not defined on these primes, ie the primes $\mathfrak{p} \in \mathrm{Proj}(k[x_{m+1}, ..., x_n])$. This is a copy of $\mathbb{P}^{n-m-1}_k$ in $\mathbb{P}^n_k$. At the level of coordinates, the two maps ι and π look the same as f and g, namely $\iota([a_0 : ... : a_m]) = [a_0 : ... : a_m : 0 : ... : 0]$ (which is a well defined closed point of $\mathbb{P}^n_k$) while $\pi([b_0 : ... : b_n]) = [b_0 : ... : b_m]$ (which is only defined on closed points of $D(x_0) \cup ... \cup D(x_m)$).

(3) *(Mathis)*

ι is the induced map from Proj of ψ. By part 2, its open subset of definition, U, is all of $\mathbb{P}^m_k$. Then by Exercise 1 of this sheet, $\iota^*\mathcal{O}_{\mathbb{P}^n_k}(1) = \mathcal{O}_{\mathbb{P}^m_k}(1)$ since ψ is homogeneous of degree $d = 1$ (note however that some elements of positive degree may map to 0, such as x_i for $i > m$).

(4) *(Mathis)*

This is exactly the same as the previous part: since π is defined exactly on $U = \mathbb{P}^n_k \setminus \mathbb{P}^{n-m-1}_k$ and that it is induced by Proj from the homogeneous degree 1 map ϕ, we obtain by Exercise 1 of this sheet that $\pi^*\mathcal{O}_{\mathbb{P}^m_k}(1) = \mathcal{O}_{\mathbb{P}^n_k}(1)|_{\mathbb{P}^n_k \setminus \mathbb{P}^{n-m-1}_k}$.

(5) *(Mathis)*

Consider the Veronese embedding $\nu : \mathbb{P}^1_k \xrightarrow{\cong} C_1 \hookrightarrow \mathbb{P}^2_k$, and write this factorisation as $\nu = \iota_1 \circ \varphi$. It is a morphism induced by Proj of a degree 2 homogeneous map so by Exercise 1 once again, we get that $\nu^*\mathcal{O}_{\mathbb{P}^2_k}(1) \cong \mathcal{O}_{\mathbb{P}^1_k}(2)$. But this is $\mathcal{O}_{\mathbb{P}^1_k}(2) \cong \varphi^*(\iota_1^*(\mathcal{O}_{\mathbb{P}^2_k}(1))) = \varphi^*(\mathcal{O}_{\mathbb{P}^2_k}(1)|_{C_1})$ as required (ie under the identification $\mathbb{P}^1_k \cong C_1$ by φ we get the result as stated in the exercise).

(6) *(Mathis)*

We identify this to a morphism $F : \mathbb{P}^1_k \to \mathbb{P}^1_k$ given by the Veronese embedding, namely $F = \pi \circ \nu$ using the notation of the previous part. It suffices to show F is finite of degree 2. F is induced by Proj for the composition $G \circ \phi$ where $G : k[x_0, x_1, x_2] \to k[x_0, x_1]$ is given by $x_0 \mapsto x_0^2, x_1 \mapsto x_1^2, x_2 \mapsto x_0x_1$. Thus F is induced by the map $k[x_0, x_1] \mapsto k[x_0, x_1]$ given by $x_0 \mapsto x_0^2$, $x_1 \mapsto x_1^2$. By Exercise 1 of sheet 3, F is thus a finite map of degree 2.

(7) *(Léo)*

First recall that the map $\pi|_{C_2} : C_2 \setminus \{P\} \to \mathbb{P}^1_k$ could be defined as the gluing of the following maps on the open cover $D_+(x_0) \cap C_2 \longrightarrow D_+(x_0)$

$$\begin{array}{cccc} \eta_0 : & k\left[\frac{x_1}{x_0}\right] & \longrightarrow & k\left[\frac{x_1}{x_0}, \frac{x_2}{x_0}\right] \Big/ \left(1 - \frac{x_1}{x_0}\frac{x_2}{x_0}\right) \\ & \frac{x_1}{x_0} & \longmapsto & \frac{x_1}{x_0} \end{array}$$

and $D_+(x_1) \cap C_2 \to D_+(x_1)$

$$\begin{array}{cccc} \eta_1 : & k\left[\frac{x_0}{x_1}\right] & \longrightarrow & k\left[\frac{x_0}{x_1}, \frac{x_2}{x_1}\right] \Big/ \left(\frac{x_0^2}{x_1^2} - \frac{x_2}{x_1}\right) \\ & \frac{x_0}{x_1} & \longmapsto & \frac{x_0}{x_1} \end{array}.$$

To extend $\pi|_{C_2}$ to C_2, we want to define a map of schemes $D_+(x_2) \cap C_2 \to D_+(x_0)$. We define it to be induced by the following map of rings

$$\begin{array}{cccc} \eta_2 : & k\left[\frac{x_1}{x_0}\right] & \longrightarrow & k\left[\frac{x_0}{x_2}, \frac{x_1}{x_2}\right] \Big/ \left(\frac{x_0^2}{x_2^2} - \frac{x_1}{x_2}\right) \\ & \frac{x_1}{x_0} & \longmapsto & \frac{x_0}{x_2} \end{array}.$$

This map will agree on intersections with $\pi|_{C_2 \setminus \{P\}}$, as $\frac{x_0}{x_2}$ is identified with $\frac{x_1}{x_0}$ by the relation $x_0^2 - x_1x_2$. By gluing all these maps together, we get an everywhere defined morphism $\pi|_{C_2} : C_2 \to \mathbb{P}^1_k$. Moreover, it is clear that such an extension is unique, as one is forced to define $\eta_2\left(\frac{x_0}{x_1}\right) = \frac{x_0}{x_2}$.

We now check that this is an isomorphism. We only need to find some open cover on which the restriction of $\pi|_{C_2}$ is an isomorphism. As $\{D_+(x_1) \cap C_2, D_+(x_2) \cap C_2\}$ is a cover of C_2 and $\{D_+(x_0), D_+(x_1)\}$ a cover of $\mathbb{P}^1_k$, it is enough to show that the restriction on these opens are isomorphisms. These restrictions will correspond to the map η_1

and η_2 previously defined. This is clear, as they both are injective and reach all the generators of their respective $k-$algebras, thanks to the quotient relation. Thus $\pi|_{C_2} : C_2 \to \mathbb{P}^1_k$ is an isomorphism.

(7) Another solution. *(Dév)*

We wish to extend the map $\pi : C_2 \setminus \mathbb{P}^0_k \to \mathbb{P}^1_k$ to all of C_2. At the ring level, this comes from the following diagram:

$$k[x_0x_1, x_0^2, x_1^2] \xleftarrow{q} k[x_0, x_1, x_2] \xrightarrow{\phi} k[x_0, x_1]$$

where q is the obvious quotient map. The leftmost ring is somewhat unwieldy, but as far as the Proj construction goes, we can add the inclusion $k[x_0x_1, x_0^2, x_1^2] \to k[x_0, x_1]$, which is an isomorphism on Proj, resulting in:

$$k[x_0, x_1] \xleftarrow{i} k[x_0x_1, x_0^2, x_1^2] \xleftarrow{q} k[x_0, x_1, x_2] \xrightarrow{\phi} k[x_0, x_1].$$

Now, upon taking Proj, the situation is best summarized by the following diagram because Proj(ϕ) isn't everywhere defined (and so the composition involving ϕ might not be so as well):

$$\begin{tikzcd}
D_+(x_0) \cong \mathbb{P}^1_k \setminus \mathbb{P}^0_k \arrow[r, hook] \arrow[d, "\sim"] & C_2 \setminus \mathbb{P}^0_k \arrow[r] \arrow[d, "\sim"] & \mathbb{P}^1_k \arrow[d, dashed, "\pi"] \\
\mathbb{P}^1_k \arrow[r, "\cong"] & C_2 \arrow[r] & \mathbb{P}^1_k
\end{tikzcd}$$

We are interested in extending the composition from $C_2 \setminus \mathbb{P}^0_k$ to $\mathbb{P}^1_k$ to all of C_2. Because the horizontal lines of the leftmost squares are isomorphisms, this is equivalent to extending the map $\mathbb{P}^1_k \setminus \mathbb{P}^0_k \to \mathbb{P}^1_k$ to all of $\mathbb{P}^1_k$. This map we wish to extend is, of course, induced by $i \circ q \circ \phi$, which we denote by α for convenience. Explicitly, this map is given by:

$$\alpha(x_0) = x_1x_0 \quad \text{and} \quad \alpha(x_1) = x_0^2.$$

Now consider the map $\beta : k[x_0, x_1] \to k[x_0, x_1]$ given by swapping x_0 and x_1, which one can think of (informally) as $\frac{\alpha}{x_0}$. Of course, the map induced by β on Proj is everywhere defined, and because α and β only differ by "multiplication by a scalar" (this is heuristics), one might hope the following diagram commutes, which would show the desired result (up to uniqueness, which we discuss at the end of this point)

$$\begin{tikzcd}
\mathbb{P}^1_k \setminus \mathbb{P}^0_k \arrow[d] \arrow[dr, "\mathrm{Proj}(\alpha)"] & \\
\mathbb{P}^1_k \arrow[r, "\mathrm{Proj}(\beta)"'] & \mathbb{P}^1_k
\end{tikzcd}$$

Indeed, it is clear that up to composition with the inverse of the isomorphism $\mathbb{P}^1_k \to C_2$, this is the desired extension, and β is clearly an isomorphism, so that we have extended it as an isomorphism.

But to check this commutativity it suffices to check that

$$\mathrm{Proj}(\alpha)\colon \mathbb{P}^1_k \setminus \mathbb{P}^0_k = D_+(x_0) \to D_+(x_1)$$

and

$$\mathrm{Proj}(\beta)_{|D_+(x_0)}\colon D_+(x_0) \to D_+(x_1)$$

are the same. But then we have the affine morphisms respectively given by

$$k[\frac{x_0}{x_1}] \to k[\frac{x_1}{x_0}] \quad \frac{x_0}{x_1} \mapsto \frac{x_1 x_0}{x_0^2} = \frac{x_1}{x_0}$$

for α, but which is therefore equal to

$$k[\frac{x_0}{x_1}] \to k[\frac{x_1}{x_0}] \quad \frac{x_0}{x_1} \mapsto \frac{x_1}{x_0}$$

the map induced by β. Note that the extension is necessarily unique because two morphisms from a reduced scheme to a separated scheme agreeing on a dense open are equal.

(8) *(Léo)*

Let $\psi_2 : \mathbb{P}^1_k \to \mathbb{P}^2_k$ be the Veronese embedding induced by the homogeneous map of rings $k[x_0, x_1, x_2] \to k[s,t]$ that sends $x_0 \mapsto st, x_1 \mapsto s^2, x_2 \mapsto t^2$. By exercise 1, we have

$$\mathcal{O}_{\mathbb{P}^2_k}(1)\Big|_{C_2} \cong {\psi_2}^* \mathcal{O}_{\mathbb{P}^2_k} \cong \mathcal{O}_{\mathbb{P}^1_k}(2)$$

and from exercise 5, we have

$$\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(2)) \cong k[s,t]_2$$

where $k[s,t]_2$ denotes the homogeneous polynomials of degree 2. It is generated as a $k-$algebra by three elements, namely s^2, t^2 and st.

On the other hand, as we know from the previous point that $\pi|_{C_2}$ is an isomorphism, we have

$$\Gamma(\mathbb{P}^1_k, (\pi\big|_{C_2})^* \mathcal{O}_{\mathbb{P}^1_k}(1)) \cong \Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(1)) \cong k[s,t]_1$$

where $k[s,t]_1$ denotes the homogeneous polynomials of degree 1, which is generated by s and t. It follows that $\mathcal{O}_{\mathbb{P}^2_k}(1)\Big|_{C_2}$ and $\left(\pi|_{C_2}\right)^* \mathcal{O}_{\mathbb{P}^1_k}(1)$ are not isomorphic, since $k[s,t]_1 \ncong k[s,t]_2$.

$\square$