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AG II - Schemes and sheaves

Exercise to hand in. *Ramifications of some self maps of* $\mathbb{P}^1$. (Due 17 November, 18:00) Please write your solution in TEX.

- We say that a map of schemes $f\colon X \to Y$ is *finite locally free* if there is a covering of Y by open affines $\mathrm{Spec}(A_i)$, with preimage $\mathrm{Spec}(B_i)$, such that induced map $A_i \to B_i$ turns B_i into a finite free A_i-module. When for every i the dimension of B_i is the same, say d, we say that the map is *finite locally free of degree d*.
- We say that a finite locally free map $X \to Y$ is *ramified at* $y \in Y$ if the geometric fiber $X_{\overline{y}}$ is not reduced.

(1) Show that the self map c_n from week 7, exercise 1 is finite locally free of degree n and identify it's ramification points.

(2) Let R be a ring. Show that the map induced on $\mathbb{P}^1_R = \mathrm{Proj}(R[x,y])$ by the R-algebra self map $x \mapsto ax+by$ and $y \mapsto cx+dy$ is an automorphism if $ad-bc \in R^\times$. We denote this map $m_{(a,b,c,d)}$.

If $R = \mathbb{C}$ and if we identify $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C}) = \mathbb{C} \cup \infty$, how is this map expressed on $\mathbb{C}$-rational points?

(3) Consider the composition

$$\mathbb{P}^1_{\mathbb{C}} \xrightarrow{c_n} \mathbb{P}^1_{\mathbb{C}} \xrightarrow{m_{(1,-1,1,1)}} \mathbb{P}^1_{\mathbb{C}} \xrightarrow{c_2} \mathbb{P}^1_{\mathbb{C}}.$$

Show it's finite locally free of fixed degree. What is the degree? What are the ramification points? Compute scheme theoretic fibers at all ramification points.

Solution key. (1) *(Kangyeon)* The preimage of $D_+(x) \cong \mathrm{Spec}(\mathbb{C}[x,y]_{(x)}) \cong \mathrm{Spec}(\mathbb{C}[t])$ (by identifying $t = y/x$) under c_n is $D_+(x^n) = D_+(x) = \mathrm{Spec}(\mathbb{C}[s])$ and the same for y. The induced morphism of $\mathbb{C}$-algebras $\mathbb{C}[t] \to \mathbb{C}[s]$ is given by $t \mapsto s^n$. Thus $\mathbb{C}[s]$ is freely generated by $1, s, \cdots, s^{n-1}$ as $\mathbb{C}[t]$-module, and the same for y. Thus c_n is locally finite free of degree n.

Let $\mathfrak{p} \in \mathbb{P}^1_{\mathbb{C}}$ be a point. If $\mathfrak{p} \in D_+(x) \cong \mathrm{Spec}(\mathbb{C}[t])$, the fiber is locally of the form $\mathrm{Spec}(\mathbb{C}[s] \otimes_{\mathbb{C}[t]} \bar{k(\mathfrak{p})})$. Since $\mathfrak{p}$ is a closed point the form $(t-\lambda)$ or the generic point (0), we compute each geometric fiber. In the first case, $k(\mathfrak{p}) = \mathbb{C}[t]/(t-\lambda) \cong \mathbb{C}$ is already algebraically closed, hence the tensor product is $\mathbb{C}[s]/(s^n - \lambda)$. If $\lambda = 0$ this is not reduced, and otherwise it is reduced, as $s^n - \lambda = \prod_{k=0}^{n-1}(s - \sqrt[n]{|\lambda|}e^{2\pi i k/n})$ is radical. For the generic point, $k(\mathfrak{p}) = \mathrm{Frac}(\mathbb{C}[t]) = \mathbb{C}(t)$, hence the tensor product is $\mathbb{C}[s] \otimes_{\mathbb{C}[t]} \bar{\mathbb{C}(t)} = (\mathbb{C}[t])[u]/(u^n - t) \otimes_{\mathbb{C}[t]} \bar{\mathbb{C}(t)} = \bar{\mathbb{C}(t)}[u]/(u^n - t) = \bar{\mathbb{C}(t)}$, which is reduced. Similarly we can deal with $D_+(y)$, and since reducedness can be checked at the level of stalks, we see that the only ramified points correspond to $\lambda = 0$ in both cases. In other identifications, they correspond to the prime ideals (x) and (y), or the points $[1 : 0] = 0$ and $[0 : 1] = \infty$.

(2) *(Kangyeon)* The R-algebra map is homogeneous of degree 1, and admits inverse if $u := ad - bc \in R^\times$. Indeed, $x \mapsto u^{-1}(dx - by)$, $y \mapsto u^{-1}(-cx + ay)$ is the inverse (which is also homogeneous of degree 1), as one can easily check by computation. Thus this surjective map induces a morphism $\mathbb{P}^1_R \to \mathbb{P}^1_R$ by the functoriality of Proj, which has inverse induced by the inverse described above. Hence this is an automorphism.

(3) *(Mathis)* We first show the following technical lemmas.

$f : X \to Y$ is finite locally free of rank d iff all induced maps of stalks $\mathcal{O}_{Y,f(\mathfrak{p})} \to \mathcal{O}_{X,\mathfrak{p}}$ make $\mathcal{O}_{X,\mathfrak{p}}$ into a free $\mathcal{O}_{Y,f(\mathfrak{p})}$ module of rank d

Proof. The $\Longrightarrow$ direction is trivial since localisations of free modules are free. For the converse, first note that f is finite locally free of rank d iff each $y \in Y$ has an affine open neighborhood U such that $f^{-1}(U) \to U$ is finite locally free of rank d. Thus wlog we may assume $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. Consider $f(\mathfrak{p}) \in Y$. We have that $B_{f(\mathfrak{p})} \to A_\mathfrak{p}$ makes $A_\mathfrak{p}$ into a free rank d module over $B_{f(\mathfrak{p})}$. Consider a basis $\{a_i/b_i\}_{i=1}^d \subset A_\mathfrak{p}$ over $B_{f(\mathfrak{p})}$. Then if we let $g = \prod_{i=1}^d b_i$, we get that A_g is free of rank d over $S^{-1}B$ with $S = f^\#(\{1, g, g^2, ...\})$. Thus if we consider $U \subset Y$ corresponding to $S^{-1}B$ about $\mathfrak{p}$, preimage contains $D(g)$, and thus up to shrinking we may assume it is contained in $D(g)$. We conclude that $f : X \to Y$ is finite locally free of rank d. □

Let $f : X \to Y$ and $g : Y \to Z$ be finite locally free maps of rank m, n respectively. Then $g \circ f$ is finite locally free of rank mn.

Proof. Follows directly from the previous lemma: a free module C of rank m on B, which is itself a free module of rank n on A, will be a free module of rank mn on A (one can also just check multiplicativity at the level of residue field extensions).

□

Note that $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$. We can now use that c_n is finite locally free of rank n, c_2 finite locally free of rank 2, and that $m_{(1,-1,1,1)}$ is finite locally free of rank 1 (since it is induced by automorphisms of rings), to deduce that $c_2 \circ m_{(1,-1,1,1)} \circ c_n$ is finite locally free of rank $2n$.

Now to compute fibers of $f = c_2 \circ m_{(1,-1,1,1)} \circ c_n$ we can note that (writing $m = m_{(1,-1,1,1)}$ for short)

$$\operatorname{Spec}(\mathbb{C}) \times_{f,\mathbb{P}^1_\mathbb{C}} \mathbb{P}^1_\mathbb{C} \cong ((\mathbb{P}^1_\mathbb{C} \times_{c_2,\mathbb{P}^1_\mathbb{C}} \operatorname{Spec}(\mathbb{C})) \times_{m,\mathbb{P}^1_\mathbb{C}} \mathbb{P}^1_\mathbb{C}) \times_{c_n,\mathbb{P}^1_\mathbb{C}} \mathbb{P}^1_\mathbb{C}$$

Thus since m is an automorphism, we can note that the points that ramify will be $c_2 \circ m([0, 1]), c_2 \circ m([1, 0])$ (since those ramify under c_n) as well $[0, 1], [1, 0]$ (since those ramify under c_2). In $\mathbb{C} \cup \infty$ this corresponds respectively to the points

$$\left(\frac{0-1}{0+1}\right)^2 = (-1)^2 = 1, \quad \left(\frac{\infty-1}{\infty+1}\right)^2 = (1)^2 = 1, \quad 0, \quad \infty$$

where we have slightly abused of notation. We should expect the generic point to be unramified. Indeed we can explicitly compute its fiber using the same computation as in exercise 1: pulling back through c_2 gives $\mathrm{Spec}(\overline{\mathbb{C}(z)}) \cup \mathrm{Spec}(\overline{\mathbb{C}(z)})$. Pulling back through m does not change the fiber structure. Finally pulling back through c_n gives $\bigsqcup_{i=1}^{2n} \mathrm{Spec}(\overline{\mathbb{C}(z)})$.

There are thus only three ramified points: $0, 1, \infty$. It remains to compute their fibers. The fibers of 0 and ∞ under c_2 are $\mathrm{Spec}(\mathbb{C}[x]/(x^2))$ up to isomorphism. We can further pull back through m and still preserve this fiber structure up to isomorphism. Now since $(c_2 \circ m)^{-1}(0)$ and $(c_2 \circ m)^{-1}(\infty)$ do not contain ramified points of c_n, we can pull back through c_n to get some $\lambda \neq 0$ such that $(\mathbb{P}^1_{\mathbb{C}})_{0,f}$ is given by Spec of

$$\mathbb{C}[x]/((x-\lambda)^2) \otimes_{\mathbb{C}[x]} \mathbb{C}[x^{1/n}] \cong \mathbb{C}[x,y]/((x-\lambda)^2, y^n - x) \cong \mathbb{C}[y]/((y^n-\lambda)^2)$$

(note how this still only has length 2). A similar thing thing holds for $(\mathbb{P}^1_{\mathbb{C}})_{\infty,f}$

For 1, this is a non ramified point of c_2 and thus its fiber is $\mathrm{Spec}(\mathbb{C}[x]/(x^2-1)) \cong \mathrm{Spec}(\mathbb{C}) \cup \mathrm{Spec}(\mathbb{C})$, corresponding to -1 and 1. Pulling further back through m preserves the fiber structure, with each copy of $\mathrm{Spec}(\mathbb{C})$ corresponding to 0 and ∞. Finally pulling back through c_n yields $\mathrm{Spec}(\mathbb{C}[x]/(x^n) \cup \mathrm{Spec}(\mathbb{C}[x]/(x^n))$.

(3) *The following calculation of fibers is more explicit. (Léo)*

Let the composition above be written $\varphi : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$, and the map of rings from which it is induced be $\tilde{\varphi} : \mathbb{C}[x,y] \to \mathbb{C}[x,y]$ sending $x \mapsto (x^n - y^n)^2$ and $y \mapsto (x^n + y^n)^2$.

We now want to find the ramification points. Let $\mathfrak{p} = (\beta x - \alpha y)$ be fixed and non-the zero ideal. Recall that $\varphi^{-1}(V_+(\mathfrak{p})) = V_+(\tilde{\varphi}(\mathfrak{p}))$[1]. The following commutative square below is a pullback

$$\begin{array}{ccc} V_+(\tilde{\varphi}(\mathfrak{p})) & \longrightarrow & \mathbb{P}^1_{\mathbb{C}} \\ \downarrow & & \downarrow \varphi \\ V_+(\mathfrak{p}) & \longrightarrow & \mathbb{P}^1_{\mathbb{C}} \end{array}$$

Moreover, one has that $V_+(\mathfrak{p}) \cong \mathrm{Spec}(k(\mathfrak{p}))$. It is therefore enough to find out if $V_+(\tilde{\varphi}(\mathfrak{p}))$ is reduced to determine wether $\mathfrak{p}$ is a ramification point or not, as one will have $\left(\mathbb{P}^1_{\mathbb{C}}\right)_{\overline{\mathfrak{p}}} \cong V_+(\tilde{\varphi}(\mathfrak{p}))$.

- $\underline{\alpha \neq \beta:}$ Moreover we assume that $\alpha, \beta \neq 0$. Without loss of generality, we may assume $\beta = 1$ and thus write $\mathfrak{p} = (x - \alpha y)$. We first show the following result.
  Claim. *One has $V_+(\tilde{\varphi}(\mathfrak{p})) \subseteq D_+(x)$.*

[1] where $\tilde{\varphi}(\mathfrak{p}))$ denotes the ideal generated by the image.

Proof. Let $\mathfrak{q} \in V_+(\tilde{\varphi}(\mathfrak{p})) \cap V_+(x)$, i.e one has

$$x,\ \tilde{\varphi}(x-\alpha y) = (1-\alpha)(x^{2n}+y^{2n}) - 2(1+\alpha)x^n y^n \in \mathfrak{q}.$$

But this implies that y^{2n} belongs to $\mathfrak{q}$ and since the latter is prime, that $y \in \mathfrak{q}$. Since $\mathfrak{q}$ does not contain the irrelevant ideal, we get a contradiction, and thus that the intersection must be empty.

□

Recall that

$$V_+(\tilde{\varphi}(\mathfrak{p})) \cong \operatorname{Proj}\left(\mathbb{C}[x,y]\big/(\tilde{\varphi}(x-\alpha y))\right)$$

as schemes. Since one also has $D_+(x) \cong \operatorname{Spec}\left(\mathbb{C}\left[\frac{y}{x}\right]\right) \cong \operatorname{Spec}(\mathbb{C}[t])$ with t sent to $\frac{y}{x}$, we get that

$$V_+(\tilde{\varphi}(\mathfrak{p})) \cong \operatorname{Spec}\left(\mathbb{C}[t]\Big/\left(t^{2n} - 2\tfrac{1+\alpha}{1-\alpha}t^n + 1\right)\right).$$

Using the Chinese remainder theorem, one can rewrite

$$\mathbb{C}[t]\Big/\left(t^{2n} - 2\tfrac{1+\alpha}{1-\alpha}t^n + 1\right) = \mathbb{C}[t]\Big/\left(t^n - \tfrac{(1+\sqrt{\alpha})^2}{1-\alpha}\right) \times \mathbb{C}[t]\Big/\left(t^n - \tfrac{(1-\sqrt{\alpha})^2}{1-\alpha}\right)$$

which will be reduced, as one can use the Chinese Remainder theorem to get that both rings are isomorphic to a product of multiple copies of $\mathbb{C}$. It follows that $\mathfrak{p} = (\beta x - \alpha y)$ is not a ramification point.

- $\underline{\alpha = \beta:}$ We have $\tilde{\varphi}(x-y) = -4x^n y^n$. Once again, one has

  $$V_+(x^n y^n) \cap V_+(x+y) = \varnothing$$

  i.e $V_+(x^n y^n) \subseteq D_+(x+y)$. Let $t := \frac{x-y}{x+y}$. We have

  $$D_+(x+y) \cong \operatorname{Spec}\left(\mathbb{C}[x,y]_{(x+y)}\right) \cong \operatorname{Spec}\left(\mathbb{C}[t]\right).$$

  With this identification, one has

  $$V_+(x^n y^n) \cong \operatorname{Spec}\left(\mathbb{C}[t]\big/(1+t)^n(1-t)^n\right).$$

  As one has

  $$\mathbb{C}[t]\big/(1+t)^n(1-t)^n \cong \mathbb{C}[t]\big/(1+t)^n \times \mathbb{C}[t]\big/(1-t)^n$$

  which is not reduced, we get that $(x-y)$ is a ramification point, whose fiber is

  $$(\mathbb{P}^1_{\mathbb{C}})_{\overline{(x-y)}} \cong \operatorname{Spec}\left(\mathbb{C}[t]\big/(1+t)^n\right) \bigsqcup \operatorname{Spec}(\left(\mathbb{C}[t]\big/(1-t)^n\right).$$

- $\underline{\alpha = 0:}$ We only do this case, as the case $\beta = 0$ is symmetric and will only result in a change of sign. Again, one has $V_+\left((x^n - y^n)^2\right) \cap V_+(x) = \varnothing$, we once again have

  $$V_+\left((x^n - y^n)^2\right) \subseteq D_+(x).$$

This gives us

$$\left(\mathbb{P}^1_{\mathbb{C}}\right)_{\overline{(x)}} = V_+\left((x^n - y^n)^2\right) \cong \operatorname{Spec}\left(\mathbb{C}[t]\Big/(1-t^n)^2\right)$$

and thus that (x) is a ramification point, as the affine scheme in the RHS is not reduced.

□