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EPFL, fall semester 2024
AG II - Schemes and sheaves

Exercise to hand in. *Basic properties of* $\mathbb{P}^n_A$. (Due Sunday October 13th, 18:00)
Please write your solution in TEX.
Let A be a ring and $A[x_0, \ldots, x_n]$ the polynomial ring in $n+1$ variables over A. We define $\mathbb{P}^n_A \coloneqq \operatorname{Proj}(A[x_0, \ldots, x_n])$.

(1) Show that $D_+(x_i)$ for $i \in \{0, \ldots n\}$ provides an open cover of $\mathbb{P}^n_A$ and that each $D_+(x_i)$ is isomorphic to $\mathbb{A}^n_A$. Hint: use the result from class about homogeneous elements of degree 1.
Note: this point is telling us that projective n-space is obtained gluing $n+1$ copies of affine n-space, in the same way it is defined using varieties in classical algebraic geometry, or in complex or real geometry.

(2) Show that $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) = A$. Hint: use the covering of the previous point and the sheaf property.
Note: this is a first instance of a more general fact about projective varieties and can be thought of as an algebraic instance of the maximum modulus principle in complex analysis.

(3) Assume that $A = k$, where k is an algebraically closed field. Show that the closed points of $\mathbb{P}^n_k$ are identified with $(n+1)$-tuples $[a_0 : \ldots : a_n]$ satisfying the following properties:
   - $a_i \in k$ for all i,
   - not all a_i are 0, and
   - two $(n+1)$-tuples $[a_0 : \ldots : a_n]$ and $[b_0 : \ldots : b_n]$ are identified if there exists $c \in k^*$ such that $b_i = c \cdot a_i$ for all i.

In other words, the points are identified with $(k^{n+1} \setminus 0)/k^\times$, *i.e.* linear subspaces of dimension 1 of k^{n+1}.

(4) Let $A = k$ be a field and B be a k-algebra. Show that every morphism of k-schemes $\mathbb{P}^n_k \to \operatorname{Spec}(B)$ is constant at the level of topological spaces with image a closed point which is k-rational. Hint: part (2) of this exercise.

(5) Let d be a positive integer and set $m \coloneqq \binom{n+d}{d} - 1$. Use Exercise 5 and monomials of degree d to define an everywhere defined morphism[1], that we call a d-th Veronese embedding of $\mathbb{P}^n_A$
$$\psi_d \colon \mathbb{P}^n_A \to \mathbb{P}^m_A.$$

(6) Let k be an algebraically closed field. Describe the image of a second Veronese embedding $\psi_2 \colon \mathbb{P}^1_k \to \mathbb{P}^2_k$. Furthermore, using part (3), describe the closed points of the image as triples $[a_0 : a_1 : a_2]$. Find a homogeneous ideal I such that $\operatorname{im}(\psi_2) = V_+(I)$ – you can use Exercise 5 for that.

[1]Induce a map using Proj from an homogeneous map of degree d that sends monomials of degree 1 to all monomials of degree d.

Solution key. (1) *(Mathis)* Let $\mathfrak{p} \in \operatorname{Proj}(A)$. Since $A[x_0, ..., x_n]_+ = (x_0, ..., x_n) \not\subset \mathfrak{p}$, there must be some x_i with $x_i \notin \mathfrak{p}$. Thus $\mathfrak{p} \subset D_+(x_i)$ so that the $D_+(x_i)$ form an open cover of $\operatorname{Proj}(A)$. Wlog it suffices now to show $D_+(x_0) \cong \mathbb{A}^n_A$ As seen in class we have an isomorphism of schemes $D_+(x_0) \cong \operatorname{Spec}(A[x_0, ..., x_n, x_0^{-1}]_0)$. We have an isomorphism $\varphi_0 : A[x_1, ..., x_n] \to A[x_0, ..., x_n, x_0^{-1}]_0$ sending $x_i \mapsto x_i/x_0$, so that $D_+(x_0) \cong \operatorname{Spec}(A[x_1, ..., x_n]) = \mathbb{A}^n_A$ as required.

(2) *(Kangyeon)* Let $s \in \Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A})$. Then the restriction $s|_{D_+(x_i)}$ is induced from $f \in \mathcal{O}_{\operatorname{Spec}(S_{(x_i)})}(\operatorname{Spec}(S_{(x_i)}) \cong S_{(x_i)} \cong A[y_1, \cdots, y_n]$ in the following way: if $f(y_1, \cdots, y_n) \in A[y_1, \cdots, y_n]$, then any $\mathfrak{p} \in D_+(x_i)$ is sent to $f(x_0/x_i, \cdots, x_{i-1}/x_i, x_{i+1}/x_i, \cdots, x_n/x_i) \in S_{(\mathfrak{p})}$. Thus it is of the form $g_i(x_0, x_1, \cdots, x_n)/x_i^{m_i}$ where $g_i \in A[x_0, x_1, \cdots, x_n]$ is homogeneous with $\deg g_i = m_i \in \mathbb{Z}_{\geq 0}$. Now for $\mathfrak{p} \in D_+(x_i) \cap D_+(x_j) = D_+(x_i x_j)$, we must have

$$\frac{g_i(x_0, x_1, \cdots, x_n)}{x_i^{m_i}} = \frac{g_j(x_0, x_1, \cdots, x_n)}{x_j^{m_j}}$$

in $S_{(\mathfrak{p})}$. As $S_{(x_i x_j)} \cong \mathcal{O}_{\operatorname{Spec}(S_{(x_i x_j)})}(\operatorname{Spec}(S_{(x_i x_j)}))$ is isomorphic to $\mathcal{O}_{\operatorname{Proj}(S)}(D_+(x_i x_j))$, we see that above fractions are equal in $S_{(x_i x_j)}$. In other words,

$$(x_i x_j)^n (g_i(x_0, x_1, \cdots, x_n) x_j^{m_j} - g_j(x_0, x_1, \cdots, x_n) x_i^{m_i}) = 0$$

in S. This forces

$$g_i(x_0, x_1, \cdots, x_n) x_j^{m_j} = g_j(x_0, x_1, \cdots, x_n) x_i^{m_i}$$

as $x_i x_j$ is not a zero-divisor, and we see that $x_i^{m_i} \mid g_i(x_0, x_1, \cdots, x_n)$. Thus by degree argument, $g_i(x_0, x_1, \cdots, x_n)/x_i^{m_i} \in A$ and they are all equal by (1), so that s is induced from $a \in A$ in a way that $s(\mathfrak{p}) = a/1 \in S_{(\mathfrak{p})}$. This map is obviously a ring homomorphism, which we have just shown to be surjective. Moreover different $a \in A$ give different element of $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A})$, for if $a/1 = 0 \in S_{(\mathfrak{p})}$ for every $\mathfrak{p} \in \operatorname{Proj}(S)$, then the annihilator I of a is not contained in any $\mathfrak{p} \in \operatorname{Proj}(S)$, so that $I \in V(S_+)$, and in particular $x_0 a = 0$ in S, which forces $a = 0$. This we have an isomorphism $A \to \Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A})$.

(3) *(Mathis)* Let $x \in \mathbb{P}^n_A$ be a closed point. x is contained in some $D^+(x_i)$ which is affine, so that x corresponds to a maximal ideal of $k[x_0, ..., \widehat{x_i}, ..., x_n] \overset{\varphi_i}{\cong} k[x_0, ..., x_n, x_i^{-1}]_0$. By weak nullstellensatz, this corresponds to an ideal $(x_0 - a_0, ..., x_n - a_n)$ for some $a_j \in k$, $j \in \{0, ..., n\} \setminus \{i\}$. This corresponds via φ_i to the homogeneous ideal $(x_0/x_i - a_0, ..., x_n/x_i - a_n)$. We thus associate the tuple $[a_0, ..., 1, ..., a_n]$ to x with the 1 in the i-th coordinate. Suppose we

associate to it another tuple $[b_0, ..., 1, ..., b_n]$ with 1 in the j-th coordinate for $x \in D^+(x_j)$. We have

$$\begin{aligned}
&\varphi_j^{-1} \circ \iota_{ji}^{-1} \circ \iota_{ij} \circ \varphi_i(x_0 - a_0, ..., x_n - a_n) = (x_0 - b_0, .., x_n - b_n) \\
&= \varphi_j^{-1} \circ \iota_{ji}^{-1}(x_0/x_i - a_0, ..., x_n/x_i - a_n) \\
&= \varphi_j^{-1} \circ \iota_{ji}^{-1}((x_0/x_j)a_j - a_0, ..., (x_n/x_j)a_j - a_n, (x_i/x_j)a_j - 1) \\
&= (a_j x_0 - a_0, ..., a_j x_n - a_n, a_j x_i - 1) \\
&= (x_0 - a_0/a_j, ..., x_n - a_n/a_j)
\end{aligned}$$

We deduce that $b_k = a_k/a_j$ (note we could divide by a_j since $x \in D^+(x_j)$ is equivalent to $a_j \neq 0$). This shows that the assignment of x to the class of $[a_0, ...1, ..., a_n]$ under multiplication by $k^\times$ is well defined (ie independent of the chosen affine containing x). By construction not all coordinates are 0 (the non-zero ones correspond to the affines in which x belongs) and they all belong to k. The assignment is injective since if x, y are assigned to the same tuple $[a_0, ..., a_n]$, then locally in some common affine $D(x_i)$ (such an affine must exist by hypothesis since the set of non-zero a_i corresponds to the affines containing x, y), x and y both correspond to the same maximal ideal so are equal. It is surjective for given a tuple $[a_0, ..., a_n]$, wlog we have $a_0 \neq 0$ so we can take a representative $[1, b_1, ..., b_n]$, and taking the closed point of $D_+(x_0)$ corresponding to $(x_1 - b_1, ..., x_n - b_n)$ gives a point associated to this tuple class

(4) *(Dév)* Using the Spec and global sections adjunction, we know that maps $\mathbb{P}^n_k \to \operatorname{Spec}(B)$ are in natural bijection with maps $B \to \mathcal{O}_{\mathbb{P}^n_k}(\mathbb{P}^n_k)$, and by point **(2)**, the global sections are isomorphic to k. Putting this together with the fact that we are working with schemes over $\operatorname{Spec}(k)$, we see that the set of maps $\mathbb{P}^n_k \to \operatorname{Spec}(B)$ is in natural bijection with section of the k-algebra structure map $k \to B$. Now we know that these are exactly the k-rational points B. Finally, using the naturality of the isomorphisms exhibiting Spec as adjoint to global sections, we get that the maps $\mathbb{P}^n_k \to \operatorname{Spec}(B)$ factor through $\operatorname{Spec}(k)$ and must map to k-rational points of B.

(5) *(Kangyeon)* There are $m+1$-number of monomials of degree d in $S = A[x_0, \cdots, x_n]$. Thus we have a homogeneous ring homomorphism of degree d $\varphi : S' = A[x_0, \cdots, x_m] \to S$ by sending each x_i to a monomial of degree d. Then for $k \geq 0$ each $S'_k \to S_{kd}$ is surjective; each monomial of degree kd can be split into k number of monomials of degree d, which are images of $x_0, \cdots, x_m$, so the original monomial is in the image of S'_k. If $k = 0$ this is trivial. Thus from this we induce a morphism of schemes from $\mathbb{P}^n_A$ to $\mathbb{P}^m_A$.

(6) *(Mathis)* We use homogeneous coordinates according to part 3 of this exercise. We pick the choice of bijection F as in the previous part to be $x_0 \mapsto x_0^2$, $x_1 \mapsto x_0 x_1$, $x_2 \mapsto x_1^2$. Let $x \in \mathbb{P}^1_k$ be described by homogeneous coordinates $[a : b]$. It thus corresponds to the homogeneous ideal $(ax_1 - bx_0)$ in $A[x_0, x_1]$. The degree 2 part of this ideal is $(ax_1^2 - bx_0x_1, ax_1x_0 - bx_0^2)$ so that it maps through ψ_2 to $(ax_2 - bx_1, ax_1 - bx_0)$ If $a \neq 0$ this is $(ax_2 - b^2/ax_0, ax_1 - bx_0)$

which corresponds to the coordinates $[a : b : b^2/a] = [a^2 : ab : b^2]$. If $b \neq 0$ we have $(ax_2 - bx_1, ax_0 - a^2/bx_1)$ which corresponds to $[a^2/b : a : b] = [a^2 : ab : b^2]$. Similarly $c \neq 0$ gives the coordinates $[a^2 : ab : b^2]$.

Now using part 4 of exercise 5, the image of ψ_2 is given by $V_+(\ker(\tilde{F}))$ (using the notation of part 5 of this problem). We claim that $\ker(\tilde{F}) = (x_0x_2 - x_1^2)$, which is a homogeneous ideal, so that we may take $I = (x_0x_2 - x_1^2)$.

Let us show the above claim. It is clear that $(x_0x_2 - x_1^2) \subset \ker(\tilde{F})$. Suppose that

$$f(x_0, x_1, x_2) = \sum_{d=0}^{\infty} f_d = \sum_{d=0}^{\infty} \sum_{a+b+c=d} \alpha_{a,b,c} x_0^a x_1^b x_2^c \in \ker(\tilde{F})$$

where the f_d are homogeneous of degree d. Since $\tilde{F}$ is homogeneous, we have that $\tilde{F}(d)$ is homogeneous of degree $2d$, and thus we must have $f_d \in \ker(\tilde{F})\ \forall d \geq 0$. In particular it is easy to see that $f_0, f_1 = 0$. For $d \geq 2$ we get then that

$$\tilde{F} f_d = \sum_{a+b+c=d} \alpha_{a,b,c} x_0^{2a+b} x_1^{2c+b} = 0$$

Now note that a monomial of the form $x_0^{2a+b} x_1^{2c+b}$ cannot be equal to $x_0^{2a'+b'} x_1^{2c'+b'}$ if $b \not\equiv b'$ mod 2. Thus we deduce that

$$\sum_{a+b+c=d, b \text{ even}} \alpha_{a,b,c} x_0^{2a+b} x_1^{2c+b} = x_0 x_1 \cdot \sum_{a+b+c=d, b \text{ odd}} \alpha_{a,b,c} x_0^{2a+b-1} x_1^{2c+b-1} = 0$$

Now note that

$$f_d + (x_0x_2 - x_1^2) = \sum_{a+b+c=d, b \text{ even}} \alpha_{a,b,c} x_0^{a+b/2} x_2^{c+b/2} + x_1 \cdot \sum_{a+b+c, b \text{ odd}} \alpha_{a,b,c} x_0^{a+(b-1)/2} x_2^{c+(b-1)/2}$$

Now the injective map $k[x_0, x_2] \to k[x_0, x_1]$ sending $x_0 \mapsto x_0^2$ and $x_2 \mapsto x_1^2$ shows that

$$\sum_{a+b+c=d, b \text{ even}} \alpha_{a,b,c} x_0^{a+b/2} x_2^{c+b/2} = \cdot \sum_{a+b+c, b \text{ odd}} \alpha_{a,b,c} x_0^{a+(b-1)/2} x_2^{c+(b-1)/2} = 0$$

and thus $f_d \in (x_0x_2 - x_1^2)$ as required

$\square$