Dr. Stefano Filipazzi
Dr. Alapan Mukhopadhyay
Léo Navarro Chafloque

EPFL, fall semester 2024
AG II - Schemes and sheaves

Exercise to hand in. *Induced map on* Spec (Due Sunday October 29, 18:00)
Please write your solution in TEX.
Let $f\colon R \to S$ be a ring homomorphism. Denote the induced map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ by $f^{\#}$.

(1) Show that the closure of the image of $f^{\#}$ is $\mathrm{V}(\ker(f))$. Find a ring theoretic property of f which is equivalent to the denseness of image of $f^{\#}$ in $\mathrm{Spec}(R)$.
(2) Find an example of a ring map where the image of the induced map on Spec is not closed. Find an example where the image is closed.
(3) Take $\mathfrak{q} \in \mathrm{Spec}(R)$. Set W to be the multiplicative subset $R \setminus \mathfrak{q}$. Prove that $(f^{\#})^{-1}(\{\mathfrak{q}\})$, as a set, is the underlying topological space of $\mathrm{Spec}(f(W)^{-1}(S/\mathfrak{q}S))$. Prove that
$$f(W)^{-1}(S/\mathfrak{q}S) \cong (\frac{R}{\mathfrak{q}})_{\mathfrak{q}} \otimes_R S$$
as rings. Here $\mathfrak{q}S$ denotes the ideal of S generated by $f(\mathfrak{q})$.
(4) Let $\iota : \mathbb{Z} \to \mathbb{Z}[X]$ be the inclusion map. Given a prime ideal $p\mathbb{Z}$ of $\mathbb{Z}$, describe $(\iota^{\#})^{-1}(p\mathbb{Z})$.
(5) Denote the algebraic closure of $\mathbb{Q}$ by $\overline{\mathbb{Q}}$. Let $\iota : \mathbb{Q}[X] \to \overline{\mathbb{Q}}[X]$ be the inclusion. Given an irreducible polynomial $g \in \mathbb{Q}[X]$, describe $(\iota^{\#})^{-1}(g\mathbb{Q}[X])$. Recall that for an irreducible polynomial g, the ideal $g\mathbb{Q}[X]$ is a prime ideal.
(6) Explain how Hilbert's Nullstellensatz gives a set-theoretic injection of $\mathbb{C}^2$ onto the closed points of $T = \mathrm{Spec}(\mathbb{C}[X, Y])$. Then, let $g \in \mathbb{C}[X, Y]$ be given. Show that there is a finite set of points $T' \subseteq T$, such that the closed points in the closure of T' in T are exactly the zeroes of g in $\mathbb{C}^2 \subseteq T$. Describe the smallest such T' in terms of g.

Solution key. 1. First we check $\overline{\mathrm{Im}(f^{\#})(\mathrm{Spec}(S))} = \mathrm{V}(\ker(f))$. For a prime ideal q of S, $f^{-1}(q)$ contains $\ker(f)$. So, $\mathrm{Im}(f^{\#})(\mathrm{Spec}(S)) \subseteq \mathrm{V}(\ker(f))$. Since $\mathrm{V}(\ker(f))$ is a closed subset, it is enough to show that the closure contains $\mathrm{V}(\ker(f))$. Consider $q \in \mathrm{V}(\ker(f))$ and a basic affine open neighborhood $\mathrm{D}(g)$ of q. Since q is prime, no powers of g is in q. So no powers of g is in $\ker(f)$. In particular, no power of $g$is zero. Thus $S[1/f(g)]$, the ring obtained by inverting powers of $f(g)$, is nonzero. So the induced map $R[1/g] \to S[1/f(g)]$ is nonzero. So the image of $f^{\#}$ contains at least a point in $\mathrm{D}(g)$. This proves q is in the closure of the image and thus our desired assertion.
We conclude that the image is dense if and only if $\mathrm{V}(\ker(f))$ is $\mathrm{Spec}(R)$. This happens if and only if $\ker(f)$ is contained in every prime ideal of R and thus in the intersection of all the prime ideals of R. Recall that the

intersection of all the prime ideals of R is the nilradical of R.

2. The image is not closed for the inclusion $\mathbb{C}[X] \to \mathbb{C}(X)$: Inclusion of the polynomial ring into its fraction field. The image is closed for the quotient map $\mathbb{C}[X] \to \mathbb{C}[X]/(x)$. **Suggestions:** *Draw a diagram of* $\mathrm{Spec}(\mathbb{C}[X])$ *and the then the image of the maps induced by the two maps above.*

3. The prime ideals of $\mathrm{Spec}(f(W)^{-1}(S/qS))$ are exactly those prime ideals of S which contain qS but are disjoint from $f(W)$ as subsets of S. Thus the inverse image of a prime ideal in $\mathrm{Spec}(f(W)^{-1}(S/qS)) \subseteq \mathrm{Spec}(S)$ under f, must contain q but cannot intersect $R \setminus q$. So the inverse image has to be q. This proves $\mathrm{Spec}(f(W)^{-1}(S/qS)) \subseteq (f^{\#})^{-1}(\{q\})$. Conversely, note that any prime ideal in $(f^{\#})^{-1}(\{q\})$ must contain qS and cannot intersect $f(W)$. The isomorphism follows from identies on tensor product relating localisation and quotients.

4. Using the description above, $(\iota^{\#})^{-1}(p\mathbb{Z})$ is

$$\mathrm{Spec}((\mathbb{Z}/p\mathbb{Z})_p[X]) \subseteq \mathrm{Spec}(\mathbb{Z}[X]).$$

When $p\mathbb{Z}$ is the zero ideal, $(\mathbb{Z}/p\mathbb{Z})_p[X] \cong \mathbb{Q}[X]$. Thus $(\iota^{\#})^{-1}(p\mathbb{Z})$ correspond to the prime ideals of $\mathbb{Q}[X]$. Since $\mathbb{Q}[X]$ is a principal ideal domain, the prime ideals are generated by irreducible polynomials. Thus $(\iota^{\#})^{-1}(p\mathbb{Z})$, in this case consist of the the prime ideals generated by irreducible polynomials of $\mathbb{Z}[X]$. When $p\mathbb{Z}$ is nonzero, the inverse image is the set of all prime ideals (p, g), where the image of q is an irreducible polynomial in $\mathbb{F}_p[X]$.

5. If g is zero, the inverse image is the zero prime ideal. If g is non-zero, the inverse image consists of the prime ideals $(X - \alpha)\overline{\mathbb{Q}}[X]$, where α belongs to the roots of g in $\overline{\mathbb{Q}}$.

6. Hilbert's Nullstellensatz implies that the maximal ideals of $\mathbb{C}[X, Y]$ are of the form $(X - a, Y - b)$, where $(a, b) \in \mathbb{C}^2$. Recall that the closed points of $\mathrm{Spec}(\mathbb{C}[X, Y])$ are the maximal ideals. Thus we get the desired set theoretic injection onto the closed points.
When g is zero, take T' to be the singletone set consisting of the zero prime ideal. Assume g is nonzero. Note that $g(a, b) = 0$ if and only if $g \in (X - a, Y - b)$. Let $g_1, g_2, \ldots, g_n$ be the non-unit irreducible factors of g. Let T' be the set of prime ideals (g_i), where i is between 1 and n. We claim that T' is desired smallest set. The closure of T' is the union of the closures of the singletone sets (g_i). So the closed points in the closure of T' is the union of the set of zeroes of g_i's in $\mathbb{C}^2$. The last union is precisely the set of zeroes of g in $\mathbb{C}^2$. Since the ideals (g_i) are radical, their zerosets are distinct by Nullstellensatz. So the closure of a proper subset of T' is strictly smaller. Since the prime ideals of $\mathbb{C}[X, Y]$ are maximal ideals or generated by an irreducible polynomial -possibly zero, we can conclude that T' is the unique smallest desired subset. Indeed such a set has to contain the generic point of each irreducible component of $V(g)$.

$\square$