SAMPLE QUESTIONS FOR THE MODERN ALGEBRAIC GEOMETRY EXAM - FALL SEMESTER OF 2020/2021

(PRELIMINARY VERSION)

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(1) Let $X = \operatorname{Spec} A$ be a Noetherian integral scheme, and let M be an A-module. Show that M is torsion-free (i.e., $0 \neq a \in A, 0 \neq m \in M \Rightarrow am \neq 0$) if and only if for every closed subset $Z \subsetneq X$ we have $\Gamma_Z\left(X, \widetilde{M}\right) = 0$.

(2) For a field k, for $X = \operatorname{Spec} k[x, y]$ and for $m = (x, y) \subseteq k[x, y]$ find the largest open set $U \subseteq X$ such that $\widetilde{m}|_U$ is locally free. Prove your claim.

(3) For an algebraically closed field k, consider the k-algebra homomorphism $\lambda : k[t] \to k[x, y, z]$ given by $t \mapsto xyz$. Which are the non-irreducible fibers of the morphism $f : \mathbb{A}^3 \to \mathbb{A}^1$ induced by λ? How many components do these fibers have? (If you need to check that a given polynomial is irreducible, explain it in the proof)

(4) Let $F : \mathbb{F}_p[t] \to \mathbb{F}_p\left[\sqrt[p]{t}\right]$ be the Fobenius (ring) homomorphism given by $t \mapsto t$ and let $f : X \to Y$ be the induced morphism of schemes. Show that $X \times_Y X$ is not reduced.

(5)
   (i) If $S = \bigoplus_{i \in \mathbb{N}} S_i$ is a graded ring generated by the degree 1 homogeneous part S_1. Let M be a graded module over S, let $p \subsetneq S_+$ be a homogeneous prime ideal of S and let $f \in S_1 \setminus p$. Show that $m \mapsto fm$ and $m \mapsto \frac{m}{f}$ induces isomorphism $(M_p)_i \to (M_p)_{i+1}$ of $(R_p)_0$ modules for every $i \in \mathbb{Z}$.

   (ii) Show that if
$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$$
is an exact sequence of graded modules over S, then the induced sequence of sheaves on $S = \operatorname{Proj} S$
$$0 \longrightarrow \widetilde{M} \longrightarrow \widetilde{N} \longrightarrow \widetilde{L} \longrightarrow 0$$
is exact.