RINGS AND MODULES 2020 - PROBLEM SHEET 1

There is one exercises in this problem sheet that will be part of the first homework. The solution has to be written in Latex and handed in as a pdf file on Moodle. The first homework is due on Sunday October 4 at 18:00. The exercise will be denoted by the symbol ** next to the exercise number.

Exercise 1. Let $R = k[x, y]$. We make $N = R \oplus R$ into a two-sided R-module via $f \cdot (p, q) = (fp, fq)$ (direct sum of R with itself).

(1) Let M be the submodule generated by the element $(x, y) \in R \oplus R$. Is $N/M \cong R$ as R-modules? Hint: R is a free R-module.
(2) Now let M be the submodule generated by the two elements $(x, 0)$ and $(0, y)$ of $R \oplus R$. Is $N/M \cong R$? Hint: Torsion

Recall that a R-module M is *simple* if the only submodules $N \subset M$ are $N = 0$ and $N = M$.

Exercise 2. **

(1) Show that any simple left R-module M is *cyclic*, i.e., isomorphic to the R-module Rm defined in the lecture, for some $m \in M$.
(2) Let M be a left R-module and let $m \in M$ be an element of M. Define $\mathcal{A}nn(m) \subset R$ to be the set of elements $r \in R$ such that $rm = 0$. Show that $\mathcal{A}nn(m)$ is a left ideal of R and that the cyclic left R-module Rm is isomorphic to the left R-module $R/\mathcal{A}nn(m)$.
Hint: Prove both statements by defining a morphism of R-modules $R \to Rm$ and investigate its kernel.
(3) Let M be a simple $k[x]$-module. Prove that $M \cong k[x]/(f)$ where f is an irreducible polynomial in $k[x]$ and (f) denotes the ideal generated by f.
(4) Which of the following $\mathbb{Z}$-modules are simple?
    (a) $\mathbb{Z}$
    (b) $\mathbb{Z}/6\mathbb{Z}$
    (c) $\mathbb{Z}/7\mathbb{Z}$

Exercise 3. Let R be a ring, M a left R-module and $m \in M$.

(a) In the previous exercise you proved that $\mathcal{A}nn(m)$ is a left ideal of R. Give an example to show that $\mathcal{A}nn(m)$ might *not* be a two sided ideal of R.

(b) Define $\mathcal{A}nn(M)$ to the set of elements $r \in R$ such that $rm = 0$ for *all* $m \in M$. Prove that $\mathcal{A}nn(M)$ is a two sided ideal of R.

Exercise 4. Let k be an algebraically closed field. In this exercise we define a non-commutative ring $\mathcal{D}(k[x]/k)$ of *differential operators* on $k[x]$ over k. The non-commutative ring $\mathcal{D}(k[x]/k)$ is a sub k-algebra (which we will abbreviate as $\mathcal{D}$) of $\mathrm{Hom}_k(k[x], k[x])$ generated by the element ∂ and x where ∂ sends a polynomial $p(x)$ to its algebraic derivative with respect to x and the element $x \in \mathrm{Hom}_k(k[x], k[x])$ is multiplication by x.

(a) Show that the following relation hold in $\mathcal{D}$: for any polynomial $P(x) \in k[x]$,

$$\partial P(x) = \frac{\partial}{\partial x} P(x) + P(x)\partial.$$

where $\frac{\partial}{\partial x} P(x)$ denotes the formal derivative of $P(x)$ with respect to the variable x. [*Hint*: Prove it by induction on the degree of P(x) and use linearity.]

(b) Show that a basis of $\mathcal{D}$ as a k-vector space is given by the elements $x^i \partial^j$, where $(i, j) \in (\mathbb{Z}_{\geq 0})^2$ if the characteristic of k is zero and $i \in \mathbb{Z}$ and $j \in \{0, 1, \ldots, p-1\}$ if the characteristic of k is $p > 0$.
[*Hint*: Use part (a) to show that an element of the form $\partial^k x^s$ can be written in terms of the proposed basis.]

(c) Now we consider a quotient of the free k-algebra on two generators. $\mathcal{D}^{form} = k\langle u, v\rangle/(uv - vu - 1)$. Show that there is a well defined ring homomorphism $\phi\colon \mathcal{D}^{form} \to End_k(k[x])$ sending $u \to \partial$ and $v \to x$. Show that ϕ is surjective onto $\mathcal{D}$ and prove that ϕ defines an isomorphism between $\mathcal{D}$ and $\mathcal{D}^{form}$ if and only if the characteristic of k is zero.

(d) Determine the submodules of $k[x]$ as a left $\mathcal{D}$-module (with left $\mathcal{D}$-module structure given by the inclusion $\mathcal{D} \subset \mathrm{End}_k(k[x])$) in the case when k has characteristic zero

(e) Show that $\mathcal{D}$ has no two sided non-trivial ideals if k has characteristic zero. I.e., show that $\mathcal{D}$ is simple.

(f) Now suppose the characteristic of k is 2. Determine the left submodules of $k[x]$ as a $\mathcal{D}$-module in this case.

Exercise 5. Let
$$0 \to M \to N \to N/M \to 0$$
be a short exact sequence of modules over a ring R. For each of the following assertions either prove that the assertion holds or provide a counterexample.

(a) If M and N/M are finitely generated, then N is too.

(b) Conversely, assume that N is finitely generated. Then N/M is finitely generated.

(c) Assume that N is finitely generated. Then M is finitely generated.

Exercise 6. (a) Let
$$0 \to M \to N \to N/M \to 0$$
be a short exact sequence of modules over a ring R.
For each of the following assertions either prove that the assertion holds or provide a counterexample.
- If N is free, then N/M is free.
- If N is free, then M is free.
- If M and N/M are free, then N is free.

(b) Let $R = \mathbb{Z}$. Is $\mathbb{Z}[x]/(x^2+1)\mathbb{Z}[x]$ a free R-module? How about $\mathbb{Z}[x]/(2x^2)\mathbb{Z}[x]$?
Is $\mathbb{Q}$ a free R-module? Is it finitely generated?

RINGS AND MODULES 2020- PROBLEM SHEET 2

There is one exercise in this problem sheet that will be part of the second homework. The solution has to be written in Latex and handed in as a pdf file on Moodle. The second homework is due on Sunday October 11 at 18:00. The exercise will be denoted by the symbol ** next to the exercise number.

Exercise 1. Answer the following questions. Provide an explanation by a proof or a counterexample.

(1) Suppose that R is a noetherian ring. Let $S \subset R$ be a subring? Is it true that S is noetherian?
(2) Let R be an Artinian ring. Is every prime ideal of R maximal?

Exercise 2. Let R be the ring of 2×2 matrices $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ such that $a \in \mathbb{Z}$ and $b, c \in \mathbb{Q}$.

(1) For each $n \in \mathbb{N}$ define $I_n = \left\{ \begin{pmatrix} 0 & \frac{m}{2^n} \\ 0 & 0 \end{pmatrix} | m \in \mathbb{Z} \right\}$. Verify that each I_n is a left ideal of R, and using the chain $I_1 \subset I_2 \subset \dots$ verify that R is not left Noetherian.
(2) Show that every right ideal of R is finitely generated, and deduce that R is right-Noetherian.

Exercise 3. Let R be a Noetherian ring. Are the following rings Noetherian? Are they Artinian?

(1) $R[x, \frac{1}{x}] := \{\sum_{i=-m}^{n} a_i x^i : a_i \in R, m, n \in \mathbb{N}\}$
(2) $R[x_1, x_2, x_3, ...]$
(3) $R[[x]]$, the ring of formal power series[1] with coefficients in R
    Hint: For each $n \in \mathbb{N}$, let $I_n := \{a_n : \sum_{i=n}^{\infty} a_i x^i \in I\}$. Then adapt the proof of the Hilbert basis theorem.
(4) $C^1(\mathbb{R})$, the ring of continuous functions $\mathbb{R} \to \mathbb{R}$ with pointwise operations.
(5) $\mathbb{R}[x]/(x-1)^2 x\mathbb{R}[x]$.

[1] $R[[x]] = \{\sum_{i=0}^{\infty} a_i x^i : a_i \in R\}$, where multiplication and addition are defined formally, as what you think they should be. These are purely formal objects: there is no requirement for any kind of convergence.

Exercise 4. ** Show that the following holds for a R-modulue M of finite length $l(M)$ (i.e., an R-modulue M that admits a composition series of finite length)

(1) If there is a short exact sequence:
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$
then $l(M) = l(M') + l(M'')$.
(2) If $N \subset M$ is a proper submodule then $l(N) < l(M)$.
(3) Use 2 to show that any strict chain of submodules in M (not necessary a maximal chain, i.e., not necessary composition series) has length $\leq l(M)$. Conclude that a module M is of finite length if and only if M is both Notherian and Artinian

Exercise 5. Let R be a ring. Let M be a finitely generated module over R and let $f : M \to M$ be an R-module homomorphism.

(1) Suppose that R is a Noetherian ring.
  (a) Does injectivity of f implies surjectivity?
  (b) Does surjectivity of f implies injectivity?
  (c) What happens if R is not necessarily Noetherian?
(2) Suppose that M is a module of finite length, show that f is injective iff f is surjective.

Exercise 6. This exercise is about *semi-simple* modules.

Definition 6.1. *A module M over a ring R is semi-simple, if it is a finite sum of its simple submodules. That is, $M = \sum_{i=1}^{d} M_i$, where $M_i \leq_R M$ are simple. A ring R is semi-simple if it is semi-simple as a left R-module.*

(1) Prove that M is semi-simple if and only if $M = \bigoplus M_i$ for some $M_i \leq_R M$ simple. I.e., prove that if $M = \sum_{i=1}^{d} M_i$ where $d \in \mathbb{N}$ is minimal with this property, then $M_i \cap M_j = 0$ for all $i \neq j$.
(2) In this exercise we prove Maschke's theorem. Let G be a finite group, and k a field such that $(|G|, \mathrm{char}(k)) = 1$. Then $k[G]$ is semi-simple.
  (a) For any ring R and any R-module M and any submodule N show that $M = N \oplus L$ for some submodule L if and only if there exists an element $\phi \in Hom_R(M, N)$ such that $\phi(n) = n$ for all $n \in N$. *Hint: Use the universal property of direct sums*
  (b) Let M be any $k[G]$-module which has finite dimension over k. Show that for any submodule N there exists an element $\phi \in Hom_{k[G]}(M, N)$ such that $\phi(n) = n$. *Hint: Take*

$\xi \in Hom_k(M, N)$ such that $\xi(n) = n$ for all $n \in N$. Show that ϕ defined by $\phi(x) = \frac{1}{|G|}\sum_{g\in G} g\xi(g^{-1}x)$ is $k[G]$-linear.

(c) Conclude the proof.

RINGS AND MODULES 2020 - PROBLEM SHEET 3

There is one exercise in this problem sheet that will be part of the third homework. The solution has to be written in Latex and handed in as a pdf file on Moodle. The third homework is due on Sunday October 18 at 18:00. The exercise will be denoted by the symbol ** next to the exercise number.

Exercise 1. Make the following computations.

(1) Compute a presentation of the $\mathbb{Z}$-module
$$M := \mathbb{Z}(2,9) + \mathbb{Z}(4,3) + \mathbb{Z}(6,8) \subseteq \mathbb{Z} \oplus \mathbb{Z}.$$

(2) Let $R = \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ be the ring of 2×2-matrices over $\mathbb{Z}$. Compute a presentation of the left R-module
$$M := R\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + R\begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \subseteq R.$$

Exercise 2. Do the following:

(1) Calculate the Smith normal form of the following matrix over $\mathbb{Z}$.
$$\begin{pmatrix} 1 & 9 & 1 \\ -2 & -6 & 0 \\ 2 & -8 & 2 \\ -1 & 1 & 5 \end{pmatrix}$$

(2) Write down the invariant factor decomposition of the $\mathbb{Z}$-module with generators e_1, e_2, e_3, e_4 and relations
$$e_1 - 2e_2 + 2e_3 - e_4 = 0$$
$$9e_1 - 6e_2 - 8e_3 + e_4 = 0$$
$$e_1 + 2e_3 + 5e_4 = 0$$

Exercise 3. ** Let $R = \mathbb{Q}[x]$. Determine the invariant factor decomposition of the R-module with generators e_1, e_2 and relations
$$x^2 e_1 + (x+1)e_2$$
$$(x^3 + 2x + 1)e_1 + (x^2 - 1)e_2$$
.

Exercise 4. Give an example of an infinitely generated $\mathbb{Z}$-module which is *not* an (infinite) direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ for various choices of n. *Hint: Revisit the last exercise on the first exercise sheet.*

Exercise 5. (1) Find a 2×2 matrix with coefficients in $\mathbb{Z}[X]$ that is not equivalent to a diagonal matrix. The equivalence that we consider here is the one introduced in the lectures, that is, up to left or right multiplication by an invertible matrix.

(2) Find also a finitely generated module over $\mathbb{Z}[X]$ that is not isomorphic to a direct sum of cyclic modules.

Exercise 6. Set $M = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and let $\alpha : \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to M$ be an isomorphism.

(1) Show that $\alpha(0 \times \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, show in general that if N is an R-module then an automorphism ϕ of N takes $\mathrm{Tors}(N)$ to $\mathrm{Tors}(N)$ bijectively.
(2) show that $\alpha(\mathbb{Z} \times 0)$ is not equal necessarily to $\mathbb{Z} \times 0$

Exercise 7. Show that an exact sequence:

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$$

of R-modules induces an exact sequence:

$$0 \longrightarrow \mathrm{Tors}(M) \longrightarrow \mathrm{Tors}(N) \longrightarrow \mathrm{Tors}(L) \ ,$$

but not necessarily an exact sequence:

$$0 \longrightarrow \mathrm{Tors}(M) \longrightarrow \mathrm{Tors}(N) \longrightarrow \mathrm{Tors}(L) \longrightarrow 0 \ .$$

PROBLEM SHEET 4 RINGS AND MODULES 2020

There is one exercise in this problem sheet that will be part of the third homework. The solution has to be written in Latex and handed in as a pdf file on Moodle. The third homework is due on Sunday October 25 at 18:00. The exercise will be denoted by the symbol ** next to the exercise number.

Exercise 1. Let $M \in \mathrm{Mat}(n, k)$ for a field k. Show that there is a basis with respect to which M is block diagonal with blocks of the form

$$\begin{pmatrix} 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & \ddots & 0 & a_1 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & a_{d-2} \\ 0 & 0 & \dots & 1 & a_{d-1} \end{pmatrix}$$

Hint: M acts naturally on some n-dimensional k-vector space V. Consider V as a $k[x]$-module via $f \cdot v = f(M)(v)$.

Exercise 2. Let R be a commutative ring, and let M be a R-module.

(1) Show that $\mathrm{Hom}_R(M, -)$ is *left exact.* That is for any exact sequence of R-modules:

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0 \ ,$$

there is an induced exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(M, N') \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(M, N'') \ .$$

(2) Give an example of a ring R and a R module M such that $\mathrm{Hom}_R(M, -)$ is not *right exact.* That is give an example of a surjection of R-modules $N \to N''$ such that the induced morphism $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N'')$ is not surjective.

Exercise 3. ** Extend the complex below to a free resolution $F_\bullet$ of the module $k := R/(x, y)$, where $R = k[x, y]$. Then compute $\mathrm{Ext}^i_{F_\bullet}(k, R)$ for each i, and note that you get the same as for the resolutions in Example 4.4.4. in the printed course notes.

$$R \oplus R \oplus R \longrightarrow R \longrightarrow k$$

The first morphism is defined by sending a basis to the following elements:

$$(1, 0, 0) \to x, (0, 1, 0) \to y, (0, 0, 1) \to x + y$$

and the second morphism is the natural surjection $R \to k$.

Exercise 4. Let $0 \to M \xrightarrow{i} Z \xrightarrow{p} N \to 0$ be an exact sequence of R-modules.

(a) A *section* of p is a morphism $s\colon N \to Z$ such that $p \circ s = id_N$. Show that p admits a section if and only if there exists an isomorphism $\Phi : Z \cong M \oplus N$ and a commuting diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] & Z \arrow[r, "p"] & N \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r, "e"] \arrow[u, equal] & M \oplus N \arrow[r, "\pi"] \arrow[u, "\Phi"] & N \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

(b) A *section* of i is a morphism $q\colon Z \to M$ such that $q \circ i = id_M$. Show that i admits a section if and only if there exists an isomorphism $\Phi : Z \cong M \oplus N$ and a commuting diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] & Z \arrow[r, "p"] & N \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r, "e"] \arrow[u, equal] & M \oplus N \arrow[r, "\pi"] \arrow[u, "\Phi"] & N \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

We say that a short exact sequence satisfying any of these conditions is split exact.

Exercise 5. Consider the ring $\mathbb{Z}[\sqrt{-5}]$.

(a) Is the ideal $(2, 1 + \sqrt{-5})$ a free $\mathbb{Z}[\sqrt{-5}]$-module?
Hint: Consider the element $6 \in \mathbb{Z}[\sqrt{-5}]$.

(b) Prove that $(2, 1 + \sqrt{-5})$ is a projective $\mathbb{Z}[\sqrt{-5}]$-module.
Hint: Prove that $(2, 1 + \sqrt{-5})$ *is projective by showing that it is a direct summand of a free module. To do this define the obvious surjection* $q : \mathbb{Z}[\sqrt{-5}]^2 \to (2, 1+\sqrt{-5})$ *and examine the assignment* $g : (2, 1 + \sqrt{-5}) \to \mathbb{Z}[\sqrt{-5}]^2$ *defined by* $g(x) = 2xe_1 - \frac{1-\sqrt{-5}}{2}xe_2$.

PROBLEM SHEET 5 RINGS AND MODULES 2020

There is one exercise in this problem sheet that will be part of the fifth homework. The solution has to be written in Latex and handed in as a pdf file on Moodle. The fifth homework is due on Sunday November 1 at 18:00. The exercise will be denoted by the symbol ** next to the exercise number.

Exercise 1. In this exercise we prove the the two *4-lemmas*. To this end, suppose that we have a commuting diagram with exact rows:

$$\begin{array}{ccccccc} A & \xrightarrow{f_1} & B & \xrightarrow{f_2} & C & \xrightarrow{f_3} & D \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} \\ A' & \xrightarrow{f_1'} & B' & \xrightarrow{f_2'} & C' & \xrightarrow{f_3'} & D' \end{array}$$

(a) Show that if a and c are epimorphisms and d is a monomorphism then b is an epimorphism.
(b) Show that if b and d are monomorphisms and a is an epimorphism then c is a monomorphism.

Exercise 2. Prove the following.
(a) If

$$0 \longrightarrow M_n \longrightarrow \dots \longrightarrow M_0 \longrightarrow 0$$

is an exact sequence of finitely generated modules over an Artinian and Notherian ring R, then $0 = (-1)^i \operatorname{length} M_i$
(b) Let $R = k[\varepsilon]$ denote (as usual) the quotient $k[x]/(x^2)$ where k is a field. Let M be the R-module $R/(x)$. Show that M has no finite resolution by finitely generated free modules.
(c) In general if R is Artinian and Noetherian, and $\operatorname{length} R \nmid \operatorname{length} M$, prove that M has no finite resolution by finitely generated free modules.
(d) Prove that over a PID every finitely generated module has a finite free resolution.

Exercise 3. Prove the following.
(a) Show that any finitely generated module over a semi-simple ring is semi-simple
(b) Show that any finitely generated module over a semi-simple ring is projective

(c) Deduce that any finitely generated module over $k[G]$ is projective, if $\operatorname{char} k \nmid |G|$

(d) What are the Ext-groups then for finitely generated $k[G]$-modules?

Exercise 4. **[1]

(a) Set $k = \mathbb{F}_p$ and $G = \mathbb{Z}/p\mathbb{Z}$. Find all the submodules (i.e. ideals) of $R = k[G]$. *Hint: Over a field of positive characteristic p we have $a^p + b^p = (a+b)^p$.*

(b) For $p = 2$, let x denote a generator of G, set $M = (x+1)$. Compute all $\operatorname{Ext}^i_R(M, M)$.

[1]as modules over $k[G]$ correspond to representations of G over k, we see that something is really wrong for $\mathbb{F}_p[Z/pZ]$ compared to the case of exercise 3.

PROBLEM SHEET 6 RINGS AND MODULES 2020

There is one exercise in this problem sheet that will be part of the sixth homework. The solution has to be written in Latex and handed in as a pdf file on Moodle. The sixth homework is due on Sunday November 8 at 18:00. The exercise will be denoted by the symbol ** next to the exercise number.

Exercise 1. In this exercise we define injective modules and prove *Baer's criterion*. We say that a left R-module Q is injective if it satisfies the following universal property:
Whenever we have a monomorphism $X \to Y$ and a homomorphism $g : X \to Q$ of left R-modules, then there exists a left R-module homomorphism $h : Y \to Q$ making the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[dl, dotted, "h"] \\ & Q & \end{tikzcd}$$

We will prove the following:

Theorem 1.1. *(Baer's Criterion) Suppose that the left R-module Q has the property that if I is any ideal of R and $f : I \to Q$ is a R-module homomorphism, there exists an R-module homomorphism $F : R \to Q$ extending f. Then Q is an injective R-module.*

We will prove *Baer's criterion* in several steps. Assume that the R-module Q satisfies Baer's criterion.

(a) Show that if $X = Ra$ and $Y = Rb$ are both *cyclic modules* and $X \to Y$ is a monomorphism and we are given a homomorphism $g : X \to Q$, then there exists a left R-module homomorphism $h : Y \to Q$ making the appropriate diagram commute. *Hint: Consider the subset of R defined by $I = \{r \in R : rb \in X\}$*

(b) Prove that if X, Y are finitely generated and we have a monomorphism $X \to Y$ and a homomorphism $g : X \to Q$ of left R-modules, then there exists a left R-module homomorphism $h : Y \to Q$ making the appropriate diagram commute. *Hint: Prove the case when $Y = X + Rb$ for some $b \notin X$ by defining the ideal I of R by $I = \{r \in R : rb \in X\}$*

(c) Use *Zorn's Lemma* to conclude the proof.

Axiom 1.2. (Zorn's Lemma)If $(\mathcal{P}, \leq)$ is a partially ordered set with the property that every totally ordered subset (often called a chain) has an upper bound, then there exists a maximal $M \in \mathcal{P}$. (that is, for $N \in \mathcal{P}$, we have $M \not\leq N$)

Exercise 2. Use Baer's Criterion to show that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

Exercise 3. ** Let $R = k[x, y]$ be the polynomial ring in two variables over an algebraically closed field k. Recall that an ideal m in a ring R is maximal if it is not properly contained in any other proper ideal of R. In this exercise you can use freely the Theorem below, which will be proven later in the course.

Theorem 3.1 (The weak Nullstellensatz in two variables)**.** *Let k be an algebraically closed field. Every maximal ideal m in the ring $k[x, y]$ is of the form $m = (x - a, y - b)$ for some $a, b \in k$.*

(a) if M is a finite length module over R, then the quotients of its composition series are of the form $R/(x - a, y - b)$.
(b) If M is a module such that $\operatorname{Ann}(M) \supseteq (x - a, y - b)$, then $\operatorname{Ann}\operatorname{Ext}^i(M, N) \supseteq (x - a, y - b)$ for every R-module N.
Hint: consider the maps $M \ni m \mapsto (x - a)m \in M$ and $M \ni m \mapsto (y - b)m \in M$. Apply then $\operatorname{Ext}^i_R(\_, N)$.
(c) Show that $\operatorname{Ext}^i(R/(x - a, y - b), N)$ is of finite length where N is any finitely generated module over R.
Hint: use the previous point
(d) Show that for each finite length module M and for each finitely generated module N over R, $\operatorname{Ext}^i_R(M, N)$ has finite length.
Hint: use the long exact sequence for a compostion series

Exercise 4. $R = k[x, y]$ as in the previous exercise (k is algebaically closed). We say that a finite length module is supported at $(x - a, y - b)$ if only $R/(x - a, y - b)$ appears as factors in the composition series. Show that if M is a finite length module supported at $(x - a, y - b)$, then $\operatorname{Ext}^i_R(M, R/(x - a', y - b')) = 0$, where $(a, b) \neq (a', b')$.

Exercise 5. For to short exact sequences:

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

and

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

we say that there is a map between them if there exists morphisms $f_i : M_i \to N_i$, for $1 \leq i \leq 3$ and a commuting diagram:

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, "f_1"] & M_2 \arrow[r] \arrow[d, "f_2"] & M_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & 0
\end{tikzcd}.$$

Show that whenever there is a map between two short exact sequences, then there is an induced map between long exact sequences of Ext-modules, making the suitable diagram commute.

Exercise 6. Show using the long exact sequence of cohomology that if $\mathrm{Ext}^1_R(M, N) = 0$, then every extension $0 \longrightarrow N \longrightarrow K \longrightarrow M \longrightarrow 0$ splits.

RINGS AND MODULES EXERCISE SHEET 7, 2020

There is one exercise in this problem sheet that will be part of the seventh homework. The solution has to be written in Latex and handed in as a pdf file on Moodle. The seventh homework is due on Sunday November 15 at 18:00. The exercise will be denoted by the symbol ** next to the exercise number.

Exercise 1. ** Let $R = k[x,y]$ and consider the R-module $M = k[x,y]/(x,y)$. Consider the free resolution:

$$0 \longrightarrow P_2 = R \xrightarrow{f_2} R \oplus R = P_1 \xrightarrow{f_1} R = P_0 \xrightarrow{f_0} M \longrightarrow 0$$

$$1 \longmapsto (y, -x)$$

$$(1,0) \longmapsto x$$

$$(0,1) \longmapsto y$$

Set $M = N$. Consider
(a) $\phi_1 : P_1 \to N$ given by $\phi_1(a,b) = f_0(a)$,
(b) $\phi_2 : P_1 \to N$ given by $\phi_1(a,b) = f_0(b)$.
Determine the isomorphism classes of the middle module of the Yoneda extension associated to $[\phi_i] \in \mathrm{Ext}^1_R(M,N)$ in Theorem 4.6.5 in the course notes.
Note: these modules are $\mathrm{coker}\left(P_1 \xrightarrow{(\phi_i, f_1)} N \oplus P_0\right)$ *for* $i = 1,2$ *as in the sequence* 6.5.i *in the above mentioned theorem, in the course notes.*

Exercise 2. Let $R = k[x,y]$.

(a) Show that $\mathrm{Ext}^1\left((x,y), R/(x,y)\right) \neq 0$.
(b) Construct a finitely generated module M such that $\mathrm{Tors}(M) \subseteq M$ is not a direct summand.

Note: $\mathrm{Tors}(M) \subseteq M$ *is always a direct summand if R is a PID by the fundamental theorem for finitely generatid modulues over PIDs.*

Exercise 3. Let R be a ring and let M, K, L and N be R-modules. Assume that $\mathrm{Ext}^i_R(M,N)$, $\mathrm{Ext}^i_R(K,N)$ and $\mathrm{Ext}^i_R(L,N)$ have finite length and that there exists integers r, s such that they are all zero for all $i < r$ and all $i > s$. Show that if

$$0 \longrightarrow K \longrightarrow M \longrightarrow L \longrightarrow 0$$

is a short exact sequence, then

$$\sum_{i=r}^{s} \operatorname{length}(-1)^i \operatorname{Ext}^i_R(M,N) =$$
$$\sum_{i=r}^{s} \operatorname{length}(-1)^i \operatorname{Ext}^i_R(K,N) + \sum_{i=r}^{s} \operatorname{length}(-1)^i \operatorname{Ext}^i_R(L,N).$$

Exercise 4. Set $R = \mathbb{Z}\left[{}^{\mathbb{Z}}\!/_{2\mathbb{Z}}\right] \cong {}^{\mathbb{Z}[x]}\!\Big/_{(x^2-1)}$. We show properties exhibiting that R is different than both $\mathbb{F}_2\left[{}^{\mathbb{Z}}\!/_{2\mathbb{Z}}\right]$ and $\mathbb{C}\left[{}^{\mathbb{Z}}\!/_{2\mathbb{Z}}\right]$:

(a) Show that R contains no simple submodules and hence show that it it is not semi-simple.

(b) Show that $R(1+x) \subseteq R$ is not projective by showing that
$$\operatorname{Ext}^1_R(R(1+x), R(1-x)) \cong {}^{R}\!/_{2R + R(1+x)} \neq 0.$$

Exercise 5. Let R be an integral domain, and let K be its fraction field.

(a) Prove that if $f \in R$ is a non-zero element, then $\operatorname{Ext}^1_R(R/(f), K) = 0$.

(b) More generally, prove that if $f_1, \dots, f_n$ is a sequence of elements such that for every $1 \leq i \leq n$ the multiplication by f_i is injective on $R/(f_1, \dots, f_{i-1})$ then
$$\operatorname{Ext}^1_R(R/(f_1, \dots, f_n), K) = 0.$$

RINGS AND MODULES 2020 - PROBLEM SHEET 8

There is one exercise in this problem sheet that will be part of the eight homework. The solution has to be written in Latex and handed in as a pdf file on Moodle. The eight homework is due on Sunday November 22 at 18:00. The exercise will be denoted by the symbol ** next to the exercise number.

Exercise 1. ** Show that $x^3 + y^7 \in k[x, y]$ is irreducible.
Hint: Use the consequence of Gauss's theorem saying that for a unique factorisation domain R and a primitive polynomial $f \in R[t]$, we have that f is irreducible in $\mathrm{Frac}(R)[t]$ *iff it is irreducible in $R[t]$*

Exercise 2. Let $R = k[x, y, z]$. Show that $(xz^3 + yz^3 - y^2z^2 + xyz - xy)$ is a prime ideal of R.
Hint: Use Eisenstein's Criterion

Exercise 3. Solve the following exercises:

(a) Consider the polynomial $f = X^3Y + X^2Y^2 + Y^3 - Y^2 - X - Y + 1$ in $\mathbb{C}[X, Y]$. Write it as an element of $(\mathbb{C}[X])[Y]$, that is collect together terms in powers of Y, and then use Eisenstein's criterion to show that f is prime in $\mathbb{C}[X, Y]$.
(b) Let F be any field. Show that the polynomial $f = X^2 + Y^2 - 1$ is irreducible in $\mathbb{F}[X, Y]$, unless F has characteristic 2. What happens in that case?

Exercise 4. Solve the following exercises:

(a) Prove that $R := \mathbb{C}[x, y, z]/(xy - z^2)$ is a domain. Calculate the transcendent degree over $\mathbb{C}$ of the fraction field of R.
(b) Calculate the dimension of the ring $\mathbb{Z}[x]$.
(c) Prove that every Artinian ring has dimension 0.
(d) Compute the dimension of the ring $\mathbb{Z}[x]/(4, x^2)$.

Exercise 5. Show the following:

(a) Prove that the only prime ideal of height zero in a domain is the ideal (0).
(b) Prove that a prime ideal of height 1 in a UFD is principal.
(c) Compute the prime ideals of height zero in $\mathbb{R}[x, y]/(xy)$.
Hint: Recall that there is a 1-1 correspondence between the prime ideals R containing I and the prime ideals of R/I.

Exercise 6. Show the following:

(a) Let $F \subset L$ be a field extension, and suppose $a_1, ..., a_n$ are elements of L which are algebraically independent over F. Prove that $F(a_1, ..., a_n)$ is isomorphic to the fraction field of the polynomial ring $F[x_1, ..., x_n]$.
(b) Let $F \subset L$ be a field extension. Show that a subset of L is a transcendence basis for L/F if and only if it is a maximal algebraically independent set. As a consequence show that a transcendence basis exists for any field extension L/F.

Exercise 7. Prove that if $F \subseteq K \subseteq L$ are field extensions such that $\operatorname{trdeg}_F L < \infty$, then $\operatorname{trdeg}_F L = \operatorname{trdeg}_F K + \operatorname{trdeg}_K L$

RINGS AND MODULES 2020
SHEET 9

There is one (sub)-exercise in this problem sheet that will be part of the ninth homework. The solution has to be written in Latex and handed in as a pdf file on Moodle. The ninht homework is due on Sunday November 29 at 18:00. The exercise will be denoted by the symbol ** next to the exercise number.

Exercise 1 (Nakayama's Lemma)**.** Let R be a ring and let M be a finitely generated R-module. Show the following:

(a) Let I be an ideal of R such that $IM = M$. Then there exists $x \in 1 + I$ such that $xM = 0$.
(b) Suppose now that the ring R is local, i.e., that there is a unique maximal ideal m of R. Suppose that $mM = M$, show that this implies that $M = 0$
(c) Show that (removing the previous assumption on R being local) that if there is an ideal $I \subset \text{nil}(R)$, where $\text{nil}(R)$ is the nilradical of R, such that $IM = M$, then this implies that $M = 0$.

Hint: Prove that in b, c the element x, whose existence is assured by a, in fact is invertible.

Exercise 2. Let R be a local ring which is an integral domain but not a field, and let F be the fraction field of R. Show that F is not finitely generated as an R-module. (After a few more lectures, you will be able to remove the assumption that R is local.)

Exercise 3. Let $\mathbb{F}_q$ be the finite field with q elements. Suppose that R is a quotient of $\mathbb{F}_q[x_1, ..., x_n]$. Prove that there is a subring $S \subset R$ such that $S \cong \mathbb{F}_q[t_1, ..., t_r]$ and R is integral over S.

Exercise 4. Let $R = \mathbb{F}_q[[t]]$ be the ring of power-series in the variable t over the field $\mathbb{F}_q$. As a set, R is the set of power-series $f = \sum_{n\in\mathbb{N}} a_n t^n$ with coefficients $a_n \in \mathbb{F}_q$. For two such power series, $\sum_{n\in\mathbb{N}} a_n t^n$ and $\sum_{n\in\mathbb{N}} b_n t^n$, one defines the addition to be the power-series $\sum_{n\in\mathbb{N}}(a_n + b_n)t^n$ and multiplication to be the power-series$\sum_{n\in\mathbb{N}}(\sum_{k=0}^{n} a_k b_{n-k})t^n$ Show the following:

(a) If $f \in R - (t)$, then f is invertible (and hence R is a local ring with maximal ideal (t)).
(b) A formal Laurent series over the field $\mathbb{F}_q$ is defined in a similar way to a formal power series, except that we also allow finitely many terms of negative degree That is series of the form $f = \sum_{n\in\mathbb{Z}} a_n t^n$ where $a_n = 0$ for all but finitely many negative indices n. Define a natural ring structure on this set and show that with this ring structure the ring of formal Laurent series over $\mathbb{F}_q$ (usually denoted $\mathbb{F}_q((t))$) is equal to the fraction field of R.
(c) Show that $\text{trdeg}_{\mathbb{F}_q}(\text{Frac}(R))$ is infinite.
Hint: show that $\mathbb{F}_q(t_1, \ldots, t_r)$ is countable, and R is not
(d) Show that $\dim R = 1$ and hence show that Thm 5.1.11 in the course notes does not work with not finitely generated algebras

Exercise 5. Show the following:

(a) If R is a domain with $\dim R = 0$, then R is a field.
(b) We say that a ring R is reduced if there are no nilpotent elements in R. I.e., if $r \in R$ is such that $r^n = 0$ then $r = 0$. Give an example of a reduced ring R of dimension zero which is not a field.

Exercise 6. ** *You should only hand in solutions to* c, d *and* e. *In proving points points* c, d *and* e *below you may freely use the results in* a, b.
Let R be an Artinian ring. Recall from Exercise 2.1 that every prime ideal of R is maximal

(a) Show that $\dim R = 0$.
(b) Show that R has finitely many maximal ideals.
Hint : for this you need the statement that if $I_1 \cap \cdots \cap I_r \subseteq p$ *for a prime ideal* $p \subseteq R$, *then* $I_i \subseteq p$ *for some* i, *which you should also show*
(c) ** There is an integer $j > 0$ such that $\text{nil}(R)^j = 0$.
Hint: Show that $nil(R)^j$ *stabilizes for* $j \gg 0$, *which we denote by* I. *In order to arrive at a contradiction assume that* $I = I^2 \neq 0$. *Consider a minimal element* J *in the set of ideals* $\{J : JI \neq 0\}$, *show that* $IJ = J$, *then show that* J *is principal. Conclude by Nakayama point c.)*
(d) **Show that if $m_1, \ldots, m_s$ are the maximal ideals of R, then $m_1^j \cdot \cdots \cdot m_s^j = 0$.
Hint: Use the statement learned in 'Anneaux et corps' that the nilardical is the intersection of all prime ideals.
(e) **Show that $\text{length}_R R < \infty$, and deduce that R is Noetherian.
Hint: construct an increasing sequence of ideals using the product of maximal ideals. Thereafter, you have to use multiple times the earlier exercise that Artinianity is closed under passage to sub- and quotient- modules.

Remark: In point (5) of Example 2.1.2 in the notes you saw an example of an Artinian module which is not Noetherian. However, the exercise above shows that an Artinian ring is always a Noetherian ring.

RINGS AND MODULES 2020
SHEET 10

There is one exercise in this problem sheet that will be part of the tenth homework. The solution has to be written in Latex and handed in as a pdf file on Moodle. The tenth homework is due on Sunday December 6 at 18:00. The exercise will be denoted by the symbol ** next to the exercise number.

Exercise 1. Let R be a ring, and M, N and P be R-modules. Show that there exists a natural bijection

$$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)).$$

Use this to prove that

$$\cdot \otimes_R M : \{R\text{-modules}\} \to \{R\text{-modules}\}, \quad A \mapsto A \otimes_R M$$

is a right exact covariant functor.

Exercise 2. ** Let R be a ring. Let M, N be R-modules and I and ideal of R. Prove that there are isomorphisms of R-modules $M \otimes_R N \cong N \otimes_R M$ and $M \otimes_R (R/I) \cong M/(IM)$.

Exercise 3. Let A be a ring, with A-algebras B and C and an A-module M. Show that:

(a) $B \otimes_A M$ naturally has the structure of a B-module,
(b) $B \otimes_A C$ naturally has the structure of an A-algebra,
(c) $B \otimes_A B$ naturally has a ring morphism to B.

Exercise 4. Prove the following assertions:

(a) Let k be a field, and let V_1 and V_2 be vector spaces over k with bases $\{e_1, ..., e_m\}$ and $\{f_1, ..., f_n\}$ respectively. Show that there is an isomorphism $V_1 \otimes_k V_2 \cong V_1^n$. In particular, show that $V_1 \otimes_k V_2$ has basis $\{e_i \otimes f_j\}$.
(b) Hence show that the element $e_1 \otimes f_2 + e_2 \otimes f_1$ cannot be written as $u \otimes v$ for any $u \in V_1$ and $v \in V_2$.

Exercise 5. Prove the following:

(a) Let R be a ring, and let I and J be two ideals such that $I + J = (1)$. Prove that $R/I \otimes_R R/J = 0$.
(b) Show that if $F \subset L$ is a field extension, $L \otimes_F L$ is a field if and only if $F = L$.

Exercise 6. Let R be a ring and M an R-module. We say that M is *flat* if for every short exact sequence of R-modules

$$0 \to A \to B \to C \to 0$$

the sequence

$$0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$$

is exact.

Prove that the following are equivalent:

(a) M is flat;
(b) $\mathrm{Tor}_i^R(A, M) = 0$ for every R-module A and every $i > 0$;

(c) $\mathrm{Tor}_1^R(A, M) = 0$ for every R-module A.

Hint: for (a)⇒(b) take a free resolution of A and tensor it with M to compute the Tor functors. For (c)⇒(a) use the long exact sequence for left derived functors.

Exercise 7. Let R be a ring.

(a) Prove that free R-modules are flat.
(b) Prove that projective R-modules are flat.
Hint: use the characterization of projective modules as direct summands of free modules
(c) Assume that R is an integral domain. Determine for which ideals I of R the R-module R/I is flat.

RINGS AND MODULES 2020
SHEET 11

There is two sub-exercises of one exercise in this problem sheet that will be part of the eleventh homework. The solution has to be written in Latex and handed in as a pdf file on Moodle. The eleventh homework is due on Sunday December 13 at 18:00. The exercise will be denoted by the symbol ** next to the exercise number.

Exercise 1. Let R be a ring containing a multiplicatively closed subset T, and let M be an R-module. Show that there is an isomorphism of R-modules:

$$T^{-1}M \cong T^{-1}R \otimes_R M.$$

Further show that this is an isomorphism of $T^{-1}R$-modules.

Exercise 2. Let R be a ring with multiplicative subset T, and suppose that L, M and N R-modules.

(a) Show that if there is an R-module homomorphism $f : M \to N$ then there is a natural $T^{-1}R$-module homomorphism $f_T : T^{-1}M \to T^{-1}N$.
(b) Show that there is an isomorphism of R-modules $T^{-1}(M \oplus N) \cong (T^{-1}M) \oplus (T^{-1}N)$.
(c) Suppose there is an exact sequence

$$0 \to L \to M \to N \to 0$$

Prove that the sequence

$$0 \to T^{-1}L \to T^{-1}M \to T^{-1}N \to 0$$

is also exact. Deduce that if $L \subset M$ is a sub R-module, then $T^{-1}(M/L) \cong (T^{-1}M)/(T^{-1}L)$ and that localization by T is an exact functor of R-modules and that $T^{-1}R$ is a flat R-module.
(d) Let p be a prime ideal of R. Show that there is an isomorphism of rings $\mathrm{Frac}(R/p) \cong R_p/(pR_p)$.
Remark: For a local ring A with maximal ideal m we call A/m the residue field of A

Exercise 3. ***You should only hand in solutions to sub-exercises a) and b)*** Let $R = F[x]$, where F is a field.

(a) ** If F is algebraically closed, then show that for every prime ideal p of R, either $R_p \cong F(x)$ or $R_p \cong F[x]_{(x)}$, where these isomorphisms are isomorphisms of F-algebras. Show that the above two cases are not isomorphic.
(b) **If $F = \mathbb{R}$, then show that up to ring isomorphism there are three possibilities for R_p, where p is a prime ideal of $F[x]$.
Hint: to tell the three cases apart, consider the residue field, to show that there are only three cases, apply linear transformations to x
(c) Show that if F is algebraically closed, then $F[x, y]$ has infinitely many prime ideals p for which $F[x, y]_p$ are pairwise non-isomorphic F-algebras. For this, you can use the following theorem of algebraic geometry:

Theorem 3.1. *For each integer $d \in \mathbb{N}\backslash\{0, 2\}$, there exist irreducible polynomials $f_d \in F[x, y]$ (of degree d) such that $\mathrm{Frac}(F[x, y]/(f_d))$ are non-isomorphic as F-algebras for different d's.*

Exercise 4. Let F be an algebraically closed field.

(a) List the prime ideals of $R = F[x, y]/(xy)$
Hint: Consider the implications of a containment $xy \in P$, for a prime ideal P. Consider the projections $R \to R/x$ and $R \to R/y$ and use that you know the prime ideals of $F[y]$ and $F[x]$.
(b) Show that for all prime ideals p of R, R_p falls into three cases up to F-algebra isomorphism, one which is a field, one which is a domain but not a field and one of which is not a domain.

Exercise 5. Let M be an A-module, and let $\mathfrak{a}$ be an ideal in A. Show that the following are equivalent:

(a) $M = 0$,
(b) $M_{\mathfrak{m}} = 0$, for every maximal ideal $\mathfrak{m}$,
(c) $M_{\mathfrak{p}} = 0$, for every prime ideal $\mathfrak{p}$.

Moreover, suppose that M is a finitely generated A-module, under this assumption prove that $M = \mathfrak{a}M$ if and only if $M_{\mathfrak{m}} = 0$ for maximal ideals satisfying $\mathfrak{a} \subset \mathfrak{m}$.

Exercise 6. Let R be a ring.

(a) Let $T \subseteq R$ a multiplicatively closed subset of R. Let q be a prime ideal of $T^{-1}R$. Let q^c be the contraction of q under $R \to T^{-1}R$. Prove that $\mathrm{ht}(q) = \mathrm{ht}(q^c)$.
(b) Let p be a prime ideal of R. Prove that $\mathrm{ht}(p) = \dim R_p$.

RINGS AND MODULES 2020
EXERCISE SHEET 12

There is one exercise in this problem sheet that will be part of the twelfth homework. The solution has to be written in Latex and handed in as a pdf file on Moodle.The twelfth homework is due on Sunday December 20 at 18:00. The exercise will be denoted by the symbol ** next to the exercise number.

Exercise 1. Let $S \to R$ be a morphism of rings. Show that a prime ideal p of S is the contraction of a prime ideal of R if and only if $p^{ec} = p$.

Hint: for one direction use ideas from the proof of "going-up" theorem

Exercise 2. Let R be a ring and $I \subset R$ be an ideal. Prove that the radical $\sqrt{I}$ of I is an ideal. Prove that if there is a containment $I \subset P \subset \sqrt{I}$ for a prime ideal P then $P = \sqrt{I}$.

Exercise 3. Let F be an algebraically closed field.

Let I, J be ideals of $R = F[x_1, ..., x_n]$. Prove that $\sqrt{I} \subset \sqrt{J}$ if and only if $\mathrm{V}(J) \subset \mathrm{V}(I)$.

Exercise 4. **Let F be an algebraically closed field. Let $R = F[x_1, ..., x_n]$ and let I and J be ideals of R. Show that

(a) $\mathrm{V}(I) \cup \mathrm{V}(J) = \mathrm{V}(I \cap J) = \mathrm{V}(IJ)$
(b) $\mathrm{V}(I) \cap \mathrm{V}(J) = \mathrm{V}(I + J)$

Exercise 5. Prove that $Z = \{(u^3, u^2v, uv^2, v^3) : u, v \in \mathbb{C}\} \subset \mathbb{C}^4$ is an algebraic set. Find $I(Z)$.

Hint: make sure you have everything!

Exercise 6. (a) Let F be an algebraically closed field, and $X \subseteq F^n$ an algebraic set with ideal $I = I(X)$. Define the coordinate ring $A(X)$ of X to be $F[x_1, \ldots, x_n]/I$. If $X = \mathrm{V}(I) \subseteq F^n$, and $Y = \mathrm{V}(J) \subseteq F^m$ are algebraic sets with $I = I(X)$ and $J = I(Y)$, then a morphism $f : X \to Y$ is defined to be a vector $(h_1, \ldots, h_m)$ of polynomials $h_i \in F[x_1, \ldots, x_n]$, such that for every $\underline{a} \in X$, $(h_1(\underline{a}), h_2(\underline{a}), \ldots, h_m(\underline{a})) \in Y$.

Show that whenever there is a morphism $f : X \to Y$ of algebraic sets as defined above there is a unique homomorphism of rings $\lambda_f : A(Y) \to A(X)$, such that the following diagram commutes.

$$\begin{array}{ccc} F[y_1,\dots,y_m] & \xrightarrow{y_i\mapsto h_i} & F[x_1,\dots,x_n] \\ \downarrow & & \downarrow \\ A(Y) & \xrightarrow{\lambda} & A(X) \end{array}$$

Here the vertical arrows are the quotient maps stemming from the definition of $A(X)$ and $A(Y)$ and the top horizontal map is given by sending y_i to $h_i(x_1, ..., x_n)$.

(b) With setup as above, show that if there is a homomorphism $\lambda : A(Y) \to A(X)$, then there is a morphism $f : X \to Y$. such that $\lambda = \lambda_f$. Furthermore, all choices of f are the same as functions from the points of X to the points of Y.

(c) Compute the integral closure R_1 of $S_1 := F[x, y]/(y^2 - x^3 - x^2)$ in the fraction field of S_1.

(d) Let R_1 and S_1 be as above. Let $S_2 := F[x, y, z]/(x^2 - y^2 z)$ and denote by R_2 the integral closure of S_2 inside its field of fractions (R_2 was computed in lectures).
For $i = 1, 2$, define the conductor ideal $\mathcal{I}_i$ to be the ideal in S_i which is the annihilator of the S_i-module R_i/S_i. Calculate $\mathcal{I}_i$ for $i = 1, 2$.

(e) With the notation as above, let $Y_i \to X_i$ be the morphisms of algebraic sets induced by the inclusion $S_i \to R_i$ for $i = 1, 2$. Assuming that $k = \mathbb{C}$, draw the real points of the X_i. Draw also in $\mathrm{V}(\mathcal{I}_i + I(X_i))$[1]. What do you notice about $\mathrm{V}(\mathcal{I}_i + I(X_i)) \subset X_i$?

Exercise 7. Let R be a ring which is the quotient of a polynomial ring over an algebraically closed field F by a radical ideal. This naturally determines an algebraic set X whose co-ordinate ring is R. Noether normalisation says there is a subring $S \subset R$ such that $S \cong F[t_1, ..., t_r]$ and R is an integral extension of S. Give a geometric interpretation of Noether normalisation. That is, the inclusion $S \to R$ corresponds to a morphism ϕ of algebraic sets. Prove that the fibres of ϕ are finite, i.e., the preimage of any point in F^r under ϕ consis of a finite set of points in X.

Exercise 8. Let F be an algebraically closed field.

[1]This is equal to the subset of X_i in F^n which is the vanishing locus of the functions in $\mathcal{I}_i$

Let X be an algebraic set in $F[x_1, ..., x_n]$ with $I(X) = I$. Prove that points of F^n contained in X are naturally in bijection with maximal ideals of $F[x_1, ..., x_n]/I$.

Exercise 9. Let F be an algebraically closed field.

Calculate the Krull dimension of the ring

$$F[w, x, y, z]/(x^2 - wy, y^2 - xz, wz - xy).$$

RINGS AND MODULES 2020
SHEET 13

Exercise 1. *Let $R = \mathbb{C}[x, y, z]$ and $I = (xy - z^2, x^2 - y^2)$. Identify $\mathrm{V}(I) \subset \mathbb{C}^3$. You should see that this naturally breaks into smaller algebraic sets. What are the ideals of each piece? How do they relate to I?*

Exercise 2. *Let F be an algebraically closed field. Let U and V be algebraic sets in F^n.*

(a) Prove that $I(U \cup V) = I(U) \cap I(V)$
(b) By considering $U = \mathrm{V}(x^2 - y)$ and $V = \mathrm{V}(y)$ for the ideals $(x^2 - y)$ and (y) in $F[x, y]$, show that it need not be true that $I(U \cap V) = I(U) + I(V)$.
(c) Prove that in general, $\sqrt{I(U) + I(V)} = I(U \cap V)$.

Exercise 3. *Let F be an algebraically closed field. Calculate a primary decomposition for the ideals:*

(a) $(x^4 - 2x^3 - 4x^2 + 2x + 3) \subset F[x]$
(b) $(x^2, xy^2) \subset F[x.y]$
(c) $(x^2, xy, xz, yz) \subset F[x, y, z]$

Exercise 4. *Let $S \subseteq R$ be a multiplicative subset and let I_i be finitely many ideals in R. By extension and contraction of ideals we shall mean extension and contraction via the natural morphism $R \to S^{-1}R$. Prove the following:*

(a) $(\bigcap_i I_i)^{ec} = \bigcap I_i^{ec}$
(b) $(\bigcap_i I_i)^e = \bigcap I_i^e$
(c) $S^{-1}(R/I) \cong S^{-1}R/I^e$, where the localization on the left is localization of an R-module
(d) If I is primary, and $u \notin \sqrt{I}$, then $(I : u) = I$
(e) For an ideal I of a ring R admitting a finite primary decomposition let $I = \cap I_i$ be such a primary decomposition, show the following:
  (a) $I^e = \bigcap_{p \supseteq I_i} I_i^e$
  (b) $I^{ec} = \bigcap_{p \supseteq I_i} I_i$
(f) From now, let $R = F[x, y]$ for a field F, $I_1 = (x)$, $I_2 = m^s$ where $m = (x, y)$ and $s > 1$ is some integer, $I_3 = (x, y - 1)^2$ and $p \subseteq R$ is a prime ideal for which we set $S = R \setminus p$.
  (a) if $p = (x)$, then $S^{-1}(R/(I_1 \cap I_2 \cap I_3)) \cong F(y)$ as an R-module

(b) *if* $p = (x, y)$, *then* $S^{-1}(R/(I_1 \cap I_2 \cap I_3)) \cong S^{-1}R/(I_1^e \cap I_2^e)$

(c) *if* $p = (x, y)$, *compute the smallest integer* n *such that* $\left(\frac{x}{1}\right)^n \in S^{-1}(R/(I_1 \cap I_2 \cap I_3))$ *is zero.*