

Modern algebraic geometry

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Chapter 1

General remarks of the course

This course is based on Hartshorne's book "Algebraic geometry". We complement this book with examples, explanation on intuition and some commutative algebra background. These notes are here for the complement, otherwise we use Hartshorne's book.

DISCLAIMER: as these notes are written this year as the course progresses, probably they will be neither perfect, nor complete. Nevertheless, we hope that it will be better to have them than not to have them.

1.1 COLOR CODES

We use color codes to distinguish certain parts. These colors serve simply to distinguish, NOT to highlight.

Review of material that was covered in "Algebraic curves"

These are parts that have been already covered in the "Algebraic curves" class.

Material very similar to what we learned in "Rings and modules"

Material that is not exactly the same, but it is very similar to something covered in the "Rings and modules" class.

1.2 LANGUAGE

These notes are written in American English, so for example we say "color" instead of "colour". Also, Hartshorne, despite of being written in American English in general, uses some French spelling of words, such as "fibre" that we do not adopt. We will just write "fiber".

Chapter 2

Review of classical algebraic geometry

This course aims to lay foundation for scheme theory, including the necessary algebra background. This is a very abstract and very general theory. For most mathematicians, it helps to understand such abstract theories and to build intuition, if there are many well understood examples to think about. For scheme theory this basic set of examples is provided by classical algebraic geometry, which is taught here at EPFL in the course “Algebraic curves”. It is essential to review the material of that course, otherwise it is really hard to understand the present course. Here we recall some of what we have learned there. We recall only statements, examples and definitions. We refer for the proofs to the “Moodle archive” page of the “Algebraic curves” course.

We also put many of the statements and definitions of “Algebraic curves” in a mildly more general setting, which is important to be able to make the connection to scheme theory. We box only those parts of the review material that is literally taken from “Algebraic curves”. The unboxed part is material that is very similar, but mildly modified, compared to what was covered in “Algebraic curves”.

In this chapter, k denotes an algebraically closed field.

2.1 AFFINE ALGEBRAIC SETS

Review of material that was covered in “Algebraic curves”

Definition 2.1.1. Denote by k^n the n -dimensional vector space over k .

An *affine algebraic set* is a subset of k^n of the following form, for some integer $n > 0$ and some ideal $I \subseteq k[x_1, \dots, x_n]$:

$$V(I) := \{ (c_1, \dots, c_n) \in k^n \mid \forall f \in I : f(c_1, \dots, c_n) = 0 \}.$$

Remark 2.1.2. Algebraic sets are the basic objects of algebraic geometry. It is not an exaggeration to say that algebraic geometry is the study of algebraic sets (there will be other types defined soon too, not only affine ones). In particular algebraic geometry is heavily connected to all the other fields where these come up. For example:

- (1) **COMPLEX DIFFERENTIAL GEOMETRY:** meaning the version of differential geometry built using domains of $\mathbb{C}^n$ and multivariable holomorphic functions as opposed to open sets of $\mathbb{R}^n$ and smooth functions (the latter is usually called real differential geometry). We note

that multivariable holomorphic functions are defined as follows (not part of the material of this course, just for the sake of completeness):

$$f = (f_1, \dots, f_m) : \mathbb{C}^n \underset{\text{open}}{\supseteq} D \rightarrow \mathbb{C}^m$$

is holomorphic if and only if so are the single variable functions

$$f_i(c_1, \dots, c_{j-1}, _, c_{j+1}, \dots, c_n) : \{x \in \mathbb{C} \mid \{c_1, \dots, c_{j-1}, x, c_{j+1}, \dots, c_n\} \in D\} \rightarrow \mathbb{C}$$

for each choices of integers $1 \leq i \leq m$, $1 \leq j \leq n$ and complex numbers $c_l \in \mathbb{C}$. In the theory of multivariable complex analysis one eventually shows that $f : D \rightarrow \mathbb{C}^m$ is holomorphic if and only if locally around each point $c \in D$, it is the value of a convergent power series around c . This maybe already hints why complex differential geometry is so close to algebraic geometry. The main difference is that algebraic geometry replaces convergent power series by polynomials. The benefit is that many of the “ugly” local behavior goes away (i.e., no essential singularities), and the theory works also over fields other than $\mathbb{C}$.

The basic object of complex differential geometry are holomorphic, or equivalently complex, manifolds. The nicest examples of these are smooth algebraic sets (we will define smoothness later). For example,

$$\{ (x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^2 + 1 = 0 \} = V((x^2 + y^2 + z^2 + 1)) \subseteq \mathbb{C}^3$$

is a smooth affine algebraic set and hence it is also a holomorphic manifold.

We will also define later in the course a notion called projective algebraic set. By a highly non-trivial theorem of Chow, over $\mathbb{C}$ projective algebraic sets in fact are the same as projective holomorphic manifolds, that is, holomorphic manifolds embedded in a projective space. This, particularly highlights how close the two theories, complex differential geometry and algebraic geometry, are. Many times these two theories consider the above same objects, however sometimes very different questions on them (e.g., questions about metrics are central to complex differential geometry, see Yau’s Fields medal, but cannot be phrased in terms of algebraic geometry).

In any case, it is a good idea to compare whatever happens in this course to real differential geometry. We would like to build a similar theory, but using polynomials instead of smooth functions. Hence, the local theory will be commutative algebra. Still, to understand the global picture the differential geometry analogy is the best.

- (2) **NUMBER THEORY:** In number theory, one considers algebraic sets that are defined over a non-algebraically closed field k_0 . The typical examples are $\mathbb{Q}$ or $\mathbb{F}_p$. Being defined over k_0 means that the generators of I of [Definition 2.1.1](#) have coefficients in k_0 . Then, in arithmetic one is mostly interested only in the k_0 -rational points, that is, those points that have all their coordinates in k_0 . For example, the probably most famous theorem of number theory, Wiles’ theorem also called the big Fermat theorem, states that if one considers the algebraic set $V((x^n + y^n - z^n)) \subseteq \overline{\mathbb{Q}}^3$ for $n \geq 3$, then it has no $\mathbb{Q}$ -rational points with (all) non-zero coordinates (here $k = \overline{\mathbb{Q}}$ denotes the algebraic closure of $k_0 = \mathbb{Q}$).

In many instances algebraic geometry can “help” the above fields. That is, by having a somewhat different perspective it can sometimes solve questions important for these fields that are much harder if not impossible to solve using their own tools.

Remark 2.1.3. Following the analogy to differential geometry explained in [Remark 2.1.2](#), affine algebraic sets are for algebraic geometry what local charts are for differential geometry. One

main difference compared to differential geometry is that in algebraic geometry local charts, that is, affine algebraic sets, can have singularities (to be precisely defined later). For example, $V((x^2 + y^2 + z^2))$ is an affine algebraic set that cannot appear as a local chart of a holomorphic manifold. In fact, if one works over $\mathbb{C}$, one can define a notion called the local fundamental group, which is $\mathbb{Z}/2\mathbb{Z}$ for $V(x^2 + y^2 + z^2)$ at $(0, 0, 0)$, but it is the group with one element for any point of any domain in $\mathbb{C}^n$. For intuition, note that the $\mathbb{Z}/2\mathbb{Z}$ comes from

$$\pi_1(V(x^2 + y^2 + z^2) \setminus \{0, 0, 0\}) = \mathbb{Z}/2\mathbb{Z}.$$

Review of material that was covered in “Algebraic curves”

Definition 2.1.4. A *Zariski closed* set of an affine algebraic set $X \subseteq k^n$ is a subset $Y \subseteq X$ which is also an affine algebraic set in k^n .

Proposition 2.1.5. *Let $X \subseteq k^n$ be an affine algebraic set. Then, the Zariski closed subsets of X form the closed sets of a topology, called the Zariski topology of X .*

Remark 2.1.6. The Zariski topology on $\mathbb{A}^1$ is the co-finite topology, that is, the open sets are exactly the complements of finite subsets. Indeed, if $Z \subseteq \mathbb{A}^1$ is a closed subset, then $Z = V(I)$ for some ideal $I \subseteq k[x]$. However, $k[x]$ is a principal ideal domain (as it is a Euclidean domain). Hence, $I = (f)$ for some $f \in k[x]$. Furthermore, as k is algebraically closed, so the irreducible elements of $k[x]$ are only the ones of the form $x - c$ for some $c \in k$. Hence, $f = \prod_{i=1}^r (x - c_i)$ for some $r \in \mathbb{Z}^{>0}$ and $c_i \in k$. Then:

$$Z = V(I) = V\left(\left(\prod_{i=1}^r (x - c_i)\right)\right) = \{c_i | 1 \leq i \leq r\}.$$

The set of all sets Z of the above form is simply the set of all the finite sets.

In particular, **WARNING:** *the Zariski topology is not Hausdorff*, as any two non-trivial open sets on $\mathbb{A}^1$ intersect non-trivially (details worked out in [Remark 2.1.7](#)). In general, an intuitive advice: the Zariski topology is probably quite different from all the topologies that you are used to.

Remark 2.1.7. **WARNING 2:** *the Zariski topology on $\mathbb{A}^2$ is not the product topology of $\mathbb{A}^1 \times \mathbb{A}^1$.*

Indeed, suppose, that $\mathbb{A}^2 \cong \mathbb{A}^1 \times \mathbb{A}^1$ as topological spaces. Then $\{(x, y) \in \mathbb{A}^2 | x \neq y\}$ is open in $\mathbb{A}^1 \times \mathbb{A}^1$. A basis of $\mathbb{A}^1 \times \mathbb{A}^1$ is

$$\{(U \times V) \subseteq \mathbb{A}^2 | U \text{ and } V \text{ are open in } \mathbb{A}^1\}$$

So, there are non-empty open sets U and V of $\mathbb{A}^1$, such that

$$U \times V \subseteq \{(x, y) \in \mathbb{A}^2 | x \neq y\}$$

That is, $U \cap V = \emptyset$. However $\mathbb{A}^1$ is an infinite set (as all algebraically closed fields are infinite just by the virtue of finite fields being not algebraically closed), and every open subset of $\mathbb{A}^1$ is cofinite. So, there can not be two disjoint open subsets of $\mathbb{A}^1$, which is a contradiction.

Review of material that was covered in “Algebraic curves”

Definition 2.1.8. If $X \subseteq k^n$ is an affine algebraic set, then the ideal of X is

$$I(X) := \{ f \in k[x_1, \dots, x_n] \mid \forall (c_1, \dots, c_n) \in X : f(c_1, \dots, c_n) = 0 \}.$$

Definition 2.1.9. If $X \subseteq \mathbb{A}^n$ is an affine algebraic set, then we define the *coordinate ring* by

$$A(X) := k[x_1, \dots, x_n] / I(X).$$

Definition 2.1.10. Let $X = V(I) \subseteq k^n$, and $Y = V(J) \subseteq k^m$ be two affine algebraic sets. A *morphism* $f : X \rightarrow Y$ is defined to be a function of sets such that there exists a vector $(h_1, \dots, h_m)$ of polynomials $h_i \in k[x_1, \dots, x_n]$ satisfying that for every $\underline{a} \in X$, $(h_1(\underline{a}), h_2(\underline{a}), \dots, h_m(\underline{a})) \in Y$.

Two morphisms $f : X \rightarrow Y$ are the same if they are the same as functions of sets. In particular, the vector $(h_1, \dots, h_m)$ is not part of the data of f .

An *isomorphism* $f : X \rightarrow Y$ is a morphism that has an inverse morphism. That is, there exists a morphism $g : Y \rightarrow X$ such that $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$.

Remark 2.1.11. Consider a morphism $f : X \rightarrow Y$ and fix also a vector $(h_1, \dots, h_m)$ as in **Definition 2.1.10**. Consider the k -algebra homomorphism $\phi : k[y_1, \dots, y_m] \xrightarrow{y_i \mapsto h_i} k[x_1, \dots, x_n]$. The homomorphism ϕ can be identified with the pullback of polynomial functions by $(h_1, \dots, h_m)$. That is, if

$$\nu(y_1, \dots, y_m) \in k[y_1, \dots, y_m] \Rightarrow \nu \circ (h_1, \dots, h_m) = \phi(\nu). \quad (2.1.a)$$

In particular,

$$\begin{array}{c} \nu \in I(Y) \Rightarrow \underbrace{\nu|_Y \equiv 0}_{\text{as a function}} \Rightarrow (\nu|_Y) \circ f \equiv 0 \Rightarrow \nu \circ (h_1, \dots, h_m)|_X \equiv 0 \Rightarrow \phi(\nu) \in I(X). \\ \uparrow \qquad \qquad \qquad \uparrow \qquad \qquad \qquad \uparrow \\ \boxed{f = (h_1, \dots, h_m)|_X \text{ by Definition 2.1.10}} \qquad \qquad \qquad \boxed{(2.1.a)} \end{array}$$

This implies that ϕ descends to a homomorphism $\lambda_f : A(Y) \rightarrow A(X)$ as pictured in the following diagram:

$$\begin{array}{ccc} k[y_1, \dots, y_m] & \xrightarrow{y_i \mapsto h_i} & k[x_1, \dots, x_n] \\ \downarrow & & \downarrow \\ A(Y) & \xrightarrow{\lambda_f} & A(X) \end{array}$$

It is a (not too hard) exercise that $(h_1, \dots, h_m)$ determines the same morphism if and only if the induced λ_f is the same (in particular, the subindex by f is reasonable), and also that any k -algebra homomorphism $A(Y) \rightarrow A(X)$ arises from a morphism $f : X \rightarrow Y$ as above. That is, we obtain a bijection

$$\boxed{\text{morphisms } f : X \rightarrow Y} \longleftrightarrow \boxed{k\text{-algebra homomorphisms } \lambda_f : A(Y) \rightarrow A(X)}$$

To sum up, we have the following statement.

Theorem 2.1.12. *There is an equivalence of categories*

$$\begin{array}{ccc} \boxed{\text{affine algebraic sets}} & \longleftrightarrow & \boxed{\text{reduced finitely generated } k\text{-algebras}} \\ \Psi \downarrow & & \downarrow \Psi \\ X & \longmapsto & A(X) \end{array}$$

Remark 2.1.13. According to [Remark 2.1.11](#), the coordinate ring is an invariant of isomorphism equivalence classes of affine varieties. In particular, we may use the coordinate ring to define an affine variety without fixing a particular closed embedding into $\mathbb{A}^n$. Furthermore, [Remark 2.1.11](#) also tells us that the evaluation of the elements of $A(X)$ at the points of X is also intrinsic. That is, if $h : X \rightarrow X$ is an isomorphism and $f \in A(X)$, then f evaluated at $h(x)$ agrees with $\lambda_h(f) = f \circ h$ evaluated at x (obviously). This then allows us to define intrinsically the following open and closed sets.

Definition 2.1.14. If X is an affine algebraic set and $f \in A(X)$, then we define

$$V_X(f) := \{ x \in X \mid f(x) = 0 \}, \text{ and } D_X(f) := \{ x \in X \mid f(x) \neq 0 \}$$

(Note: if one fixes a particular representation of X as a closed subset of $\mathbb{A}^n$, and considers the induced homomorphism $\phi : k[x_1, \dots, x_n] \rightarrow A(X)$, then $V_X(f) = V(\phi^{-1}((f)))$).

If $I \subseteq A(X)$ is an ideal, we define

$$V_X(I) := \{ x \in X \mid \forall f \in I : f(x) = 0 \}.$$

If $Y \subseteq X$ is a closed subset, then we define

$$I_X(Y) := \{ f \in A(X) \mid f|_Y \equiv 0 \}.$$

Sometimes, if it is obvious what variety we work on, we omit the subindex X from the above notations.

Review of material that was covered in “Algebraic curves”

Proposition 2.1.15. [NULLSTELLENSATZ FOR COORDINATE RINGS] *Let X be an affine algebraic set. Then $V(I)$ and $I_X(Y)$ of [Definition 2.1.14](#) yield bijections as follows*

$$\boxed{\text{prime ideals } I \subseteq A(X)} \longleftrightarrow \boxed{Z \subseteq X \text{ irreducible closed subsets}}$$

$$\boxed{\text{radical ideals } I \subseteq A(X)} \longleftrightarrow \boxed{Z \subseteq X \text{ closed subsets}}$$

and

$$\boxed{\text{maximal ideals } m \subseteq A(X)} \longleftrightarrow \boxed{x \in X}.$$

In particular, $I(V(I)) = \text{rad}(I)$ for any ideal $I \subseteq A(X)$.

Definition 2.1.16. A *quasi-affine algebraic set* is a an open subset $U \subseteq X$ in an affine algebraic set. The (Zariski) topology on it is the subspace topology induced from the topology on X , or equivalently from the topology on $\mathbb{A}^n$.

2.2 NOETHERIAN TOPOLOGICAL SPACE - IRREDUCIBILITY

Definition 2.2.1. A topological space is Noetherian, if all descending chains of closed subsets

$$Z_1 \supseteq Z_2 \supseteq Z_3 \supseteq \dots$$

stabilize eventually.

Remark 2.2.2. The usual euclidean topology is almost never Noetherian. For example, in $\mathbb{R}$ the closed sets $[0, \frac{1}{2^n}]$ (for $n \in \mathbb{Z}^{>0}$) form an infinite decreasing chain.

Review of material that was covered in “Algebraic curves”

Definition 2.2.3. A topological space Z is irreducible, if whenever $Z_1 \cup Z_2 = Z$ for some closed subsets $Z_i \subsetneq Z$, then $Z_i = Z$ holds for some i .

Lemma 2.2.4. If X is a Noetherian topological space, and Y is a closed subset, then the subspace topology on Y is also Noetherian.

Proposition 2.2.5. Every closed subset $Z \subseteq X$ of a Noetherian topological space can be written as $Z = \bigcup_i Z_i$ for finitely many irreducible closed subsets, which decomposition is unique if we assume that it is minimal (i.e., non of the Z_i can be dropped).

Definition 2.2.6. An *affine variety* is an irreducible affine algebraic set.

2.3 REGULAR FUNCTIONS AND MORPHISMS OF QUASI-AFFINE VARIETIES

Review of material that was covered in “Algebraic curves”

Notation 2.3.1. If X is an affine variety, the field of rational functions is defined by $K(X) := \text{Frac}(A(X))$.

Remark 2.3.2. Recall that according to [Remark 2.1.13](#), the coordinate ring $A(X)$ is associated intrinsically to every affine variety X (which means precisely that it is invariant under isomorphism). Furthermore, elements of $A(X)$ can be evaluated, again intrinsically, at points of X .

Using the previous paragraph, elements of $K(X)$ can be thought of as functions on some open set of X . To explain this, fix $f \in K(X)$. Choose also a particular representation $f = \frac{a}{b}$. Then f can be regarded as a usual function $D(b) \rightarrow k$, where $D(b) = D_X(b)$ is the open set defined in [Definition 2.1.14](#). Indeed, at every $x \in D(b)$, $\frac{a(x)}{b(x)}$ yields a unique element of k , and furthermore if we have a different representation, say $f = \frac{c}{d}$ of f , then by the identity $ad = bc$, the evaluation gives the same element of k : $a(x)d(x) = b(x)c(x) \Rightarrow \frac{c(x)}{d(x)} = \frac{a(x)}{b(x)}$. This way we may even associate a domain to each $f \in K(X)$, which is phrased in the next definition.

Definition 2.3.3. If X is an affine variety and $f \in K(X)$, then the *domain* $\text{Dom}(f)$ of f is the largest open set $U \subseteq X$ on which f can be regarded as a function $U \rightarrow k$. By the discussion of [Remark 2.3.2](#):

$$\text{Dom}(f) = \bigcup_{f=\frac{a}{b}} D(b).$$

Remark 2.3.4. However, before going there, we note that not only a rational function f yields an actual functions $U \rightarrow k$ on some open set U , but different rational functions yield different actual functions on an non-empty open set $U \subseteq X$. In fact we have the following:

Claim. If $f, g \in K(X)$ and $\emptyset \neq U \subseteq X$ is an open set such that $U \subseteq \text{Dom}(f)$, $U \subseteq \text{Dom}(g)$, and the functions $U \rightarrow k$ associated to f and g are equal, then $f = g$.

Proof. Choose a point $x \in U$. By [Definition 2.3.3](#), $f = \frac{a}{b}$ and $g = \frac{c}{d}$ for some $a, b, c, d \in A(X)$, with $b(x) \neq 0$ and $d(x) \neq 0$. By shrinking U around x , we may also assume that $U \subseteq D(b)$ and $U \subseteq D(d)$. For the statement of the claim, we need to prove that $ad = bc$ as elements of $A(X)$, or equivalently that $ad - bc = 0$. By our assumptions, as a function over U , $ad - bc|_U \equiv 0$. Hence, $U \subseteq V(ad - bc)$. As U is dense by the irreducibility of X , this implies that $V(ad - bc) = X$. Then, by [Proposition 2.1.15](#) we obtain that $ad - bc = 0$. $\square$

Definition 2.3.5. Let X be an affine variety, and let $\emptyset \neq U \subseteq X$ be an open set. Then the *ring of regular functions* $\mathcal{O}_X(U) \subseteq K(X)$ on U is the following subring ((easy) homework to verify closedness under multiplication and addition):

$$\{ f \in K(X) \mid U \subseteq \text{Dom}(f) \}$$

We define $\mathcal{O}_X(\emptyset) = \{0\}$. We treat $\mathcal{O}_X(U)$ as an algebra over k .

Remark 2.3.6. For non-irreducible affine algebraic sets, the definition of $\mathcal{O}_X(U)$ is very similar. Let X be sitting inside $\mathbb{A}^n$. Then we define $\mathcal{O}_X(U)$ to be the ring of functions $s : U \rightarrow k$ which satisfy the following: for each $(x_1, \dots, x_n) \in U$, there are $f, g \in k[x_1, \dots, x_n]$ such that $f(x_1, \dots, x_n)/g(x_1, \dots, x_n) = s(x_1, \dots, x_n)$.

One reason why this is harder than the above definition for the case of varieties is that one has to verify that it is independent of the embedding in $\mathbb{A}^n$. Also, this does not give a subring of something (in our case this something was $K(X)$), and hence the restriction morphisms are more intricate than in the irreducible case (where restriction of functions agrees with inclusions given by the subring structures).

Example 2.3.7. The above notions are easily computable if $A(X)$ is a UFD. Indeed, then we may choose a unique representative $f = \frac{a}{b}$ (up to multiplication by a unit) such that $\gcd(a, b) = 1$. We claim that for every $f = \frac{a'}{b'}$ there is a $c \in A(X)$ such that $a' = ca$ and $b' = cb$. Indeed, $ab' = ba'$ can hold by the relative prime assumption only if $b|b'$ and $a|a'$. Then it follows that $b'/b = a'/a$, which we then call c . This finishes the proof of our claim.

The above claim implies that, if $A(X)$ is a UFD, $D(b') \subseteq D(b)$ for every representative $f = \frac{a'}{b'}$, and hence by [Definition 2.3.3](#), $D(f) = D(b)$. In particular,

$$\mathcal{O}_X(U) = \left\{ \frac{a}{b} \in K(X) \mid U \subseteq D(b) \right\} = S^{-1}A(X),$$

for $S = \{b \in A(X) \mid U \subseteq D(b)\}$ (note that this is indeed a multiplicatively closed set).

Some more explicit examples are the following.

- (1) If $X = \mathbb{A}^1$, $U = \mathbb{A}^1 \setminus \{0\}$, then

$$\mathcal{O}_X(U) \cong k[x]_x \cong k[x, x^{-1}].$$

(2) If $X = \mathbb{A}^1$, $U = \mathbb{A}^1 \setminus \{0, 1\}$, then

$$\mathcal{O}_X(U) \cong k[x, x^{-1}, (x-1)^{-1}] \cong k[x]_{x(x-1)}.$$

(3) If $X = \mathbb{A}^2$, $U = \mathbb{A}^2 \setminus V(x_1)$, then

$$\mathcal{O}_X(U) \cong k[x_1, x_2]_{x_1} \cong k[x_1, x_2, x_1^{-1}].$$

However, if $A(X)$ is not a UFD, then whether $f \in K(X)$ is contained $\mathcal{O}_X(U)$ cannot be determined by looking at a single representative. For example, if we take $y(y-x)-x(x-1)(x+1)$ (which is irreducible because of Eisenstein's criterion applied in $k[x][y]$ with $p = x$), then for $f = \frac{(x+1)x}{y} = \frac{y-x}{(x-1)}$ the first representation shows that $(1, 1) \in \text{Dom}(f)$ (and the second does not do it), while the second representation shows that $(-1, 0)$ and $(0, 0)$ are in $\text{Dom}(f)$ (and the first representation does not do it). Then, using a Riemann-Roch computation (something we will learn at the very end of this course) one can show that there is no representative $\frac{a}{b} = f$ such that both of the above representatives can be obtained by multiplying both the numerator and the denominator of $\frac{a}{b}$ by $c \in A(X)$.

Proposition 2.3.8. *If X is an affine variety, then $\mathcal{O}_X(X) = A(X)$, as subrings of $K(X)$.*

Proof. Trivially $A(X) \subseteq \mathcal{O}_X(X)$, as for $\frac{a}{1}$ the denominator is non-zero at every point $x \in X$.

So, take $f \in \mathcal{O}_X(X) \subseteq K(X)$. By definition, for each point $x \in X$, $f = \frac{a_x}{b_x}$, such that $b_x \in U_x := D(b_x)$. As $\bigcup_{x \in X} U_x = X$, we have $\emptyset = \bigcap_{x \in X} V(b_x) = V((b_x | x \in X))$, which is equivalent to saying that $(b_x | x \in X) = (1)$ according to Proposition 2.1.15. In particular, there is $x_1, \dots, x_s \in X$, and $r_1, \dots, r_s \in A(X)$, such that $\sum_{i=1}^s r_i b_{x_i} = 1$. Set $U_i := U_{x_i}$, $a_i := a_{x_i}$ and $b_i := b_{x_i}$. Then for any fixed $1 \leq j \leq s$, we have the following stream of equalities in $K(X)$.

$$f = \frac{a_j}{b_j} \underset{\boxed{\sum_{i=1}^s r_i b_i = 1}}{=} \frac{\sum_{i=1}^s r_i b_i a_j}{b_j} \underset{\boxed{b_i a_j = a_i b_j}}{=} \frac{\sum_{i=1}^s r_i a_i b_j}{b_j} = \sum_{i=1}^s r_i a_i \in A(X).$$

This concludes our proof. $\square$

So, indeed for an affine algebraic set X , the collection of all $\mathcal{O}_X(U)$ contains the information about $A(X)$. Furthermore, it opens the door to define morphisms also for quasi-affine varieties.

Definition 2.3.9. If $U \subseteq \mathbb{A}^n$ is a quasi-affine variety, then by definition it is an open set in the closure $\overline{U}$, where $\overline{U}$ is an affine variety. Then, we define $K(U) := K(\overline{U})$, and similarly we define $\mathcal{O}_U$ the following way: for each open set $V \subseteq U$, $\mathcal{O}_U(V) := \mathcal{O}_{\overline{U}}(V)$.

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Definition 2.3.10. Let X and Y be two quasi-affine varieties. Then $f : X \rightarrow Y$ is a *morphism* if it is a continuous map such that for each open set $V \subseteq Y$, and $s \in \mathcal{O}_Y(V)$, $s \circ (f|_{f^{-1}V}) \in \mathcal{O}_X(f^{-1}V)$ (as a function). By abuse of notation, most of the time we write $s \circ f$ instead of $s \circ (f|_{f^{-1}V})$.

An *isomorphism* $f : X \rightarrow Y$, as in the case of affine varieties, is a morphism that has an inverse morphism. That is, there exists a morphism $g : Y \rightarrow X$ such that

$g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$.

Remark 2.3.11. As before for affine varieties, a consequence of the introduction of morphisms for quasi-affine varieties is that from now we identify isomorphic ones, and hence we forget the fixed embedding. Also, from now we should treat only those notions for quasi-affine varieties that are invariant under isomorphisms.

Proposition 2.3.12. *For affine varieties, the above definition coincides with the one we gave earlier in Definition 2.1.10.*

Proposition 2.3.13. *If X is quasi-affine, then $\mathcal{O}_X$ are invariant under isomorphism. That is, if $f : X \rightarrow Y$ is an isomorphism, and $V \subseteq Y$ is an open set, then $\mathcal{O}_X(f^{-1}V) \cong \mathcal{O}_Y(V)$ via precomposition by f .*

Proposition 2.3.14. *If X is quasi-affine, we have*

$$K(X) = \bigcup_{U \subseteq X \text{ open}} \mathcal{O}_X(U).$$

In particular, an isomorphism $f : X \rightarrow Y$ of quasi-affine varieties yields isomorphism $f^\# : K(Y) \rightarrow K(X)$ given by $g \mapsto g \circ f$ for any $g \in \mathcal{O}_X(U)$.

Remark 2.3.15. It is hard to overestimate the theoretical importance of Proposition 2.3.13. It tells us that the data of $\mathcal{O}_X(U)$ for all open set $U \subseteq X$ describes uniquely the isomorphism equivalence class of a variety, which is the point of view of modern algebraic geometry (scheme theory).

Notation 2.3.16. From now *affine variety* (resp. *quasi-affine variety*) means a variety isomorphic to an affine variety (resp. a quasi-affine variety).

Proposition 2.3.17. *If X is an affine variety and $f \in A(X)$, then $D(f)$ is also affine with $A(D(f)) \cong A(X)_f$.*

2.4 PROJECTIVE ALGEBRAIC SETS

Review of material that was covered in “Algebraic curves”

Definition 2.4.1. The projective space $\mathbb{P}_k^n$ of dimension n over k is defined as a set as follows:

$$\mathbb{P}^n := k^{n+1} \setminus (0, \dots, 0) / k^\times,$$

that is the set of non-zero $n+1$ -tuples $(a_0, \dots, a_n) \in k^{n+1}$, where any $(\lambda a_0, \dots, \lambda a_n)$ and $(a_0, \dots, a_n)$ are identified for any $n+1$ -tuple $(a_0, \dots, a_n)$ and any $\lambda \in k^\times := k \setminus \{0\}$. Usually we denote by $[a_0, \dots, a_n]$ the equivalence class via the above identification.

Remark 2.4.2. $\mathbb{P}^n$ can be thought of as the space of lines in k^{n+1} through the origin.

Remark 2.4.3. $\mathbb{P}^n$ can be also thought of as adding one point at infinity to $\mathbb{A}^n$ for each line in $\mathbb{A}^n$. This represents the “limit point” as one “approaches infinity” along the given line. We will explain this more after defining the standard affine patches.

Our next task is to define projective algebraic sets. Intuitively, these are the “compactifications” of affine algebraic sets by adding the infinity limit points. This approach can be made completely precise over $\mathbb{C}$ by considering the usual Euclidean (non Zariski) topology. However, over general algebraically closed fields one has to go around, define first projective algebraic sets without referring to the above limiting property, and then it can be realized that projective algebraic sets are in fact closures in Zariski topology of affine algebraic sets.

Definition 2.4.4. A *graded ring* is a ring R , for which the additive group $(R, +)$ decomposes as a direct sum

$$R = \bigoplus_{i=0}^{\infty} R_i,$$

and for which the multiplicative ring operations respect this decomposition. That is:

- (1) $1 \in R_0$, and
- (2) $R_i \cdot R_j \subseteq R_{i+j}$.

The standard structure of a graded ring on $k[x_0, \dots, x_n]$ (or the standard grading) is via the degree. That is, $k[x_0, \dots, x_n]_i$ is the additive group of homogeneous polynomials of degree i .

An ideal $I \subseteq R$ in a graded ring is homogeneous if $I = \bigoplus_{i=0}^{\infty} I_i$, where $I_i \subseteq R_i$ are additive subgroups.

Remark 2.4.5. Note that I_i in Definition 2.4.4 are unique, as one has to have $I_i := I \cap R_i$.

Remark 2.4.6. If $f_1, \dots, f_r \in k[x_0, \dots, x_n]$ are homogeneous, then $(f_1, \dots, f_r)$ is a homogeneous ideal.

Conversely, if $I \subseteq k[x_0, \dots, x_n]$ is a homogeneous ideal, then it can be generated by (finitely many) homogeneous elements $f_1, \dots, f_r \in k[x_0, \dots, x_n]$. Indeed, to prove this, by the Noetherianity of $k[x_1, \dots, x_n]$, it is enough to show that if $f_1, \dots, f_r \in I$ are homogeneous elements and $(f_1, \dots, f_r) \subsetneq I$, then there is a homogeneous $f_{r+1} \in I \setminus (f_1, \dots, f_r)$. However, as both $(f_1, \dots, f_r) \subsetneq I$ are homogeneous ideals, Remark 2.4.5 implies that there is a value of i for which $(f_1, \dots, f_r) \cap k[x_0, \dots, x_n]_i \subsetneq I \cap k[x_0, \dots, x_n]_i$.

Example 2.4.7. Following Remark 2.4.6, $I = (x^2, y)$ is a homogeneous ideal. However, one has to be careful: not all generators of homogeneous ideals are homogeneous. For example, $I = (x^2 + y, y)$ is a non-homogeneous generator set of the homogeneous ideal I .

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Definition 2.4.8. A *projective algebraic set* is a subset of $\mathbb{P}_k^n$ of the form

$$V(I) := \{ [c_0, \dots, c_n] \in \mathbb{P}_k^n \mid \forall j \in \mathbb{N}, \forall f \in I_j : f(c_0, \dots, c_n) = 0 \}.$$

for some homogeneous ideal $I \subseteq k[x_0, \dots, x_n]$.

Remark 2.4.9. Since elements of $\mathbb{P}_k^n$ are equivalence classes of $n+1$ -tuples, for the above definition being sensible, it is essential to verify that $f(c_0, \dots, c_n)$ being zero is independent of the choice of element in the equivalence class $[c_0, \dots, c_n]$. That is, we have to verify that for all $\lambda \in k^\times$:

$$f(c_0, \dots, c_n) = 0 \Leftrightarrow f(\lambda c_0, \dots, \lambda c_n) = 0.$$

This follows from f being homogeneous of degree j , so:

$$\begin{aligned} f(\lambda c_0, \dots, \lambda c_n) &= \sum_{\sum_{s=0}^n i_s = j} a_{i_0, \dots, i_n} (\lambda c_0)^{i_0} \dots (\lambda c_n)^{i_n} \\ &= \sum_{\sum_{s=0}^n i_s = j} a_{i_0, \dots, i_n} \lambda^{\sum_{s=0}^n i_s} c_0^{i_0} \dots c_n^{i_n} = \lambda^j f(c_0, \dots, c_n). \end{aligned}$$

Definition 2.4.10. The subsets $Y \subseteq X \subseteq \mathbb{P}_k^n$ of a projective algebraic set X that are themselves projective algebraic sets in $\mathbb{P}_k^n$ are called the Zariski closed subsets of X . It is an easy exercise that this defines indeed a topology, and that this is the subspace topology induced from the topology on $\mathbb{P}^n$.

Definition 2.4.11. A *quasi-projective* algebraic set is a Zariski open subset $U \subseteq X$ of a projective algebraic set $X \subseteq \mathbb{P}^n$, endowed with the subspace topology inherited from $\mathbb{P}^n$, or equivalently from X .

Definition 2.4.12. For any integer $i \in \{0, 1, \dots, n\}$, the i -th standard open chart of $\mathbb{P}^n$ is defined to be the open subset $\mathbb{P}^n \setminus V(x_i) = \{ [x_0, \dots, x_n] \in \mathbb{P}^n \mid x_i \neq 0 \}$. We identify this open set with $\mathbb{A}^n$ via the bijective assignment, see [Proposition 2.4.13](#), which in one direction is

$$\phi_i : \mathbb{P}^n \setminus V(x_i) \ni [x_0, \dots, x_n] \mapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right)$$

and in the other direction is

$$\xi_i : (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \mapsto [x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n].$$

Proposition 2.4.13. In the situation of [Definition 2.4.12](#), ϕ_i and ξ_i are inverses of each other.

Remark 2.4.14. The above identification of $\mathbb{P}^n \setminus V(x_i)$ with $\mathbb{A}^n$ corresponds geometrically to putting a hyperplane at $x_i = 1$, and sending each line (thinking about $\mathbb{P}^n$ as the space of lines in k^{n+1} through the origin) to the intersection point with this hyperplane.

Definition 2.4.15. In the situation of [Definition 2.4.12](#) we define the following.

- (1) For any element $f \in k[x_0, \dots, x_n]$ the *dehomogenization* $f^\flat$ of f is defined as follows (which is an abuse of notation as f is not assumed to be homogeneous here):

$$f^\flat := f(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) \in k[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n].$$

For any homogeneous ideal $I \subseteq k[x_0, \dots, x_n]$, we define the *dehomogenization* $I^\flat$ of I to be the following ideal of $k[x_1, \dots, x_n]$:

$$I^\flat := \{ f^\flat \mid f \in I \}.$$

- (2) For any $g \in k[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$ we define its *homogenization* $g^\flat$ to be the following element of $k[x_0, \dots, x_n]$:

$$g^\flat := x_i^{\deg g} \cdot g\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right),$$

where the equality takes place in $k(x_0, \dots, x_n)$, which by definition is the fraction field of $k[x_0, \dots, x_n]$. Equivalently, if $g = \sum_{j=0}^{\deg g} g_j$ is the decomposition into homogeneous parts, then

$$g^{\mathfrak{h}} = \sum_{j=0}^{\deg g} x_i^{\deg g - j} g_j.$$

For an ideal $J \subseteq k[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$, we define the *homogenization* of J to be

$$J^{\mathfrak{h}} := \left(g^{\mathfrak{h}} \mid g \in J \right),$$

where parentheses mean the ideal generated by the given elements.

Remark 2.4.16. In [Definition 2.4.15](#), we defined dehomogenization also for non-homogeneous elements of $k[x_0, \dots, x_n]$ as a homogeneous ideal I contains also non-homogeneous elements, and we wanted to be able to make the definition $I^{\mathfrak{d}} = \{f^{\mathfrak{d}} \mid f \in I\}$.

Remark 2.4.17. Dehomogenization behaves better than homogenization. That is, dehomogenization yields an algebra homomorphism. On the other hand, homogenization does not even depend on the monomials of a polynomial $g(x)$, but also on $\deg g$. Hence, the same monomial $m(x)$ in two polynomials $g_1(x)$ and $g_2(x)$ of different degrees contribute with differently to the homogenizations $g_1^{\mathfrak{h}}$ and $g_2^{\mathfrak{h}}$.

Lemma 2.4.18. *In the situation of [Definition 2.4.15](#), and of [Definition 2.4.12](#),*

- (1) $\phi_i(V(I) \cap U_i) = V(I^{\mathfrak{d}})$
- (2) $\xi_i(V(J)) = V(J^{\mathfrak{h}}) \cap U_i$

Proposition 2.4.19. *If $X \subseteq \mathbb{P}^n$ is a projective algebraic set, then $X \setminus V(x_i)$ is homeomorphic to an affine algebraic set if regarded as a subset of $\mathbb{A}^n$ via the correspondence $\mathbb{P}^n \setminus V(x_i) \leftrightarrow \mathbb{A}^n$ of [Definition 2.4.12](#).*

Remark 2.4.20. Let us explain the necessity of [Proposition 2.4.19](#). Recall that the topology on X was defined by using the Zariski topology on $\mathbb{P}^n$ that is, by homogeneous polynomials in variables $x_0, \dots, x_n$. In the meanwhile the topology on $\mathbb{A}^n$ is defined by using non-homogeneous polynomials in almost the same variables, that is, we throw away x_i . So, if one identifies $X \setminus V(x_0)$ with a closed algebraic subset of $\mathbb{A}^n$ via the correspondence of [Definition 2.4.12](#), one needs to also match up these two topologies.

Now, we would like to define morphisms also for quasi-projective varieties. We use verbatim the definition given for quasi-affine varieties ([Definition 2.3.10](#)), but we need to define then $\mathcal{O}_X(U)$ also for X quasi-projective. The idea is based on the observations of [Remark 2.4.24](#).

Definition 2.4.21. If $X \subseteq \mathbb{P}^n$ is a projective algebraic set, then the *homogeneous coordinate ring* is defined to be $S(X) := k[x_0, \dots, x_n] / I(X)$. As I is homogeneous, $S(X)$ is automatically a graded ring, with setting $S(X)_d = k[x_0, \dots, x_n]_d / I_d$.

Remark 2.4.22. As we have seen in [Remark 2.4.9](#), the values of an $f \in k[x_0, \dots, x_{n+1}]$ are not well defined at the points of $\mathbb{P}^n$, however its vanishing locus is well defined. This yields the same for the elements of $S(X)$. Hence, we may perform the following definitions, mirroring the affine case done in [Definition 2.1.14](#).

Definition 2.4.23. If $X \subseteq \mathbb{P}^n$ is a projective algebraic set and $f \in S(X)$ homogeneous, then we define

$$D_X(f) = \{ x \in X \mid f(x) \neq 0 \} \quad \text{and} \quad V_X(f) = \{ x \in X \mid f(x) = 0 \}.$$

Remark 2.4.24. Similarly to the affine case, if $X \subseteq \mathbb{P}^n$ is a projective variety, then, as $I(X)$ is a prime ideal, $S(X)$ is a domain. Hence, we may consider its function field $\text{Frac}(S(X))$. Let $\text{Frac}(S(X))_{\text{hom}}$ be the subset of $\text{Frac}(S(X))$, consisting of elements $f \in \text{Frac}(S(X))$ which can be written as a fraction $\frac{a}{b}$ of homogeneous elements of the same degree d .

Note that elements $f \in \text{Frac}(S(X))_{\text{hom}}$ can be regarded as usual functions on adequate open sets. Indeed, if we write $f = \frac{a}{b}$ as above, then although neither $a(x)$ and $b(x)$ are well defined at any point $x \in D(b)$ (a significant difference to the affine case), multiplication by a scalar λ multiplies both $a(x)$ and $b(x)$ by λ^d . This cancels when taking fraction, and then $\frac{a(x)}{b(x)}$ becomes well defined on the representatives of $x \in \mathbb{P}^n$.

Furthermore, consider different representation of the same $f \in \text{Frac}(S(X))_{\text{hom}}$. So, let $f = \frac{a}{b}$ and $f = \frac{c}{d}$, where $a, b, c, d \in S(X)$ are homogeneous, and $\deg a = \deg b$ and $\deg c = \deg d$. In this case, by the identity $ad = bc$ the evaluation gives the same element of k . Hence we may regard f also as a function $f : U \rightarrow k$ for any open set U contained in $\bigcup_{f=\frac{a}{b}} D(b)$.

Additionally, the argument of [Remark 2.3.4](#) can be moved to the projective case literally, yielding that if $f \neq g \in \text{Frac}(S(X))_{\text{hom}}$, such that both f and g yield functions on some non-empty open set U , then these functions are different. So, we may indeed think about elements of $\text{Frac}(S(X))$ as functions on adequate open sets.

Remark 2.4.25. Let $X \subseteq \mathbb{P}^n$ be a projective variety, and choose $f \in K(X)$ that yields a function $\tilde{f} : U \rightarrow k$ on some non-empty open set of X , as explained in [Remark 2.4.24](#). Consider now $Y_i \subseteq \mathbb{A}^n$, the affine variety homeomorphic to $D(x_i) \cap X$ via the affine charts of [Definition 2.4.12](#). Let $\eta : X \cap D(x_i) \rightarrow Y_i$ be this homeomorphism. Then, we may hope that considering $\tilde{f}|_{D(x_i) \cap U}$ as a function on $\eta(D(x_i) \cap U)$, is a regular function on $\eta(D(x_i) \cap U)$. This is in fact true, and this also yields a description of $K(X)$ in terms of homogeneous polynomials. All of this is phrased below in [Proposition 2.4.26](#).

Proposition 2.4.26. *Let $X \subseteq \mathbb{P}^n$ be a projective variety, and for any integer $0 \leq i \leq n$ such that $X \cap D(x_i) \neq \emptyset$, let Y_i be the affine variety homeomorphic to $X \cap D(x_i)$ via [Definition 2.4.12](#). The restriction map of functions described in [Remark 2.4.25](#) yields an isomorphism:*

$$\xi : \text{Frac}(S(X))_{\text{hom}} := \left\{ \frac{f}{g} \in \text{Frac}(S(X)) \mid \exists d \geq 0 : f, g \in S(X)_d \right\} \rightarrow K(X \cap D(x_i))$$

Algebraically, this isomorphism is given by the following two homomorphisms:

(1)

$$\text{Frac}(S(X)) \ni \frac{f}{g} \mapsto \frac{[\bar{f}^{\mathfrak{d}}]}{[\bar{g}^{\mathfrak{d}}]} \in \text{Frac}(A(Y_i)),$$

where

- f and g are required to be homogeneous of the same degree,
- $\bar{f}$ and $\bar{g}$ are homogeneous lifts of f and g to $k[x_0, \dots, x_n]$, respectively,
- $\bar{f}^{\mathfrak{d}}$ and $\bar{g}^{\mathfrak{d}}$ are the dehomogenization with respect to x_i of $\bar{f}$ and $\bar{g}$, respectively, see [Definition 2.4.15](#), and
- $[\bar{f}^{\mathfrak{d}}]$ and $[\bar{g}^{\mathfrak{d}}]$ are the residue classes of $\bar{f}^{\mathfrak{d}}$ and $\bar{g}^{\mathfrak{d}}$ in $A(Y_i)$, respectively.

(2)

$$\text{Frac}(A(Y_i)) \ni \frac{f}{g} \mapsto \frac{[\tilde{f}_h]}{[\tilde{g}_h]} \in \text{Frac}(S(X)),$$

where

- $\tilde{f}$ and $\tilde{g}$ are lifts of f and g , respectively, to $k[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$,

- $\tilde{f}_h$ and $\tilde{g}_h$ homogenizations of $\tilde{f}$ and $\tilde{g}$ of the same degree, that is, by setting $m := \max\{\deg f, \deg g\}$, and using the notations of [Definition 2.4.15](#),

$$\tilde{f}_h := x_i^{m-\deg f} \tilde{f}^h, \text{ and } \tilde{g}_h := x_i^{m-\deg g} \tilde{g}^h.$$

- $[\tilde{f}_h]$ and $[\tilde{g}_h]$ are residue classes of $\tilde{f}_h$ and $\tilde{g}_h$ in $S(X)$.

Definition 2.4.27. If $X \subseteq \mathbb{P}^n$ is a quasi-projective variety, we define the *field of rational functions* of X as

$$K(X) := \left\{ \frac{f}{g} \in \text{Frac}(S(\overline{X})) \mid \exists d \geq 0 : f, g \in S(\overline{X})_d \right\},$$

where $\overline{X}$ is the closure of X in $\mathbb{P}^n$.

If $s \in K(X)$, then the *domain* $D(s)$ of s is the largest open set U on which s can be represented as a function $U \rightarrow k$, that is, by [Remark 2.4.24](#):

$$\text{Dom}(s) = \bigcup_{s = \frac{f}{g}, f, g \in S(\overline{X})_d} X \cap D_{\overline{X}}(g).$$

Remark 2.4.28. Note that, in the situation of [Definition 2.4.27](#), if $U \subseteq X$, then $\mathcal{O}_X(U) = \mathcal{O}_{\overline{X}}(U)$ and $K(X) = K(\overline{X})$. So, [Definition 2.4.27](#) is compatible with the philosophy of [Definition 2.3.9](#).

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Just as in the affine case:

Definition 2.4.29. If $X \subseteq \mathbb{P}^n$ is a quasi-projective variety, and $U \subseteq X$ is an open set, then the ring $\mathcal{O}_X(U)$ of *regular functions on U* is the following subring of $K(X)$:

$$\{ s \in K(X) \mid U \subseteq \text{Dom}(s) \} \subseteq K(X).$$

Then the definition for a morphism in the quasi-projective case is just as in the quasi-affine case verbatim.

Definition 2.4.30. Let X and Y be two algebraic sets, where any of the two can be either quasi-projective (in which case we still have fixed embeddings $X \subseteq \mathbb{P}^n$ or $Y \subseteq \mathbb{P}^m$), or quasi-affine. Then $f : X \rightarrow Y$ is a *morphism* if it is a continuous map such that for each open set $V \subseteq Y$, and $s \in \mathcal{O}_Y(V)$, $s \circ (f|_{f^{-1}V}) \in \mathcal{O}_X(f^{-1}V)$. Just as in the quasi-affine case, by abuse of notation, we usually simply write $s \circ f$ for $s \circ (f|_{f^{-1}V})$.

A morphism is an *isomorphism* if it has an inverse.

Remark 2.4.31. Just as in the case of quasi-affine varieties (show in [Proposition 2.3.13](#) and [Proposition 2.3.14](#)), $\mathcal{O}_X$ and $K(X)$ are invariant under isomorphisms also for quasi-projective varieties.

Furthermore, just as in the affine case, [Definition 2.4.30](#) removes the data of the fixed embedding to $\mathbb{P}^n$ from the notion of quasi-projective varieties.

[Proposition 2.4.26](#) tells us that the homeomorphisms stated in [Proposition 2.4.19](#) are in fact isomorphisms:

Proposition 2.4.32. Let X be a quasi-projective variety and let Y be the quasi-affine variety homeomorphic to $D(x_i) \cap X$ via the affine chart maps given by [Definition 2.4.12](#) (for some embedding $X \subseteq \mathbb{P}^n$ and for some choice of i). Then, the homeomorphism between $D(x_i) \cap X$ and Y is in fact an isomorphism.

Notation 2.4.33. From now on, a variety X being projective/affine/quasi-affine means that X is isomorphic to a projective/affine/quasi-affine variety.

Corollary 2.4.34. *Quasi-affine varieties are quasi-projective.*

Corollary 2.4.35. *Each quasi-projective variety can be covered by finitely many affine open sets.*

Remark 2.4.36. It is again hard to overestimate the importance of [Corollary 2.4.35](#) in algebraic geometry. It tells us that the local structure of varieties is given by affine varieties. So, affine varieties play the role of what “smooth charts” play in differential geometry. And, indeed, many arguments in algebraic geometry localize to the affine case via [Corollary 2.4.35](#).

Example 2.4.37. As earlier in [Example 2.3.7](#) it is easy to compute $\mathcal{O}_X(U)$, if X is projective, and $S(X)$ is a UFD, taking into account that the divisors of a homogeneous polynomial are also homogeneous. So, if $f \in S(X)$ is a fraction of homogeneous polynomials of the same degree, we may put it into a form $\frac{a}{b}$, where a and b are relatively prime and homogeneous of the same degree (and they are then uniquely defined up to multiplication by a unit). As in [Example 2.3.7](#), we obtain that in this situation $f \in \mathcal{O}_X(U)$ if and only if $U \subseteq D(b)$.

So, for example $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k$, as b in this case can be only a constant function (since otherwise $V(b) \neq \emptyset$), and then a also has to be a constant function, as $\deg a = \deg b$ must hold.

It follows then the following corollary.

Corollary 2.4.38. $\mathbb{P}^n$ is not affine for $n > 0$.

Proof. We have seen in the above example that $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k$, but if it was affine, then it would be $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = A(\mathbb{P}^n)$, which would be a ring of dimension $n > 0$. However, $\dim k = 0$. $\square$

2.5 LOCAL RINGS

Definition 2.5.1. Let X be a variety. Let $V \subseteq X$ be an irreducible closed subvariety. Then the *local ring* of X at V is defined by

$$\mathcal{O}_{X,V} := \{ f \in K(X) \mid \text{Dom}(f) \cap V \neq \emptyset \}.$$

If $V = \{x\}$ for a single point $x \in X$, then we use the notation $\mathcal{O}_{X,x}$ for $\mathcal{O}_{X,\{x\}}$.

Remark 2.5.2. In the case $V = \{x\}$, the condition $\text{Dom}(f) \cap V \neq \emptyset$ is equivalent to the condition $x \in \text{Dom}(f)$. So, $\mathcal{O}_{X,x} = \{ f \in K(X) \mid x \in \text{Dom}(f) \}$.

Remark 2.5.3. The definition of $\mathcal{O}_{X,V}$ given in [Definition 2.5.1](#) is equivalent (by [Definition 2.4.29](#)) to the other form given in some of the books:

$$\bigcup_{U \subseteq X \text{ open}, U \cap V \neq \emptyset} \mathcal{O}_X(U) \subseteq K(X)$$

Lemma 2.5.4. THE LOCAL RING IS INDEED A RING.: *If $V \subseteq X$ is a closed subvariety of a variety, then $\mathcal{O}_{X,V}$ is a subring of $K(X)$.*

Example 2.5.5. [Lemma 2.5.4](#) fails if V is not irreducible. Indeed, set $X := \mathbb{A}_{x,y}^2$ and $V := V(xy)$, $f := \frac{1}{x}$ and $g := \frac{1}{y}$. Note that for $s \in A(X)$, we have $V(s) \cap \text{Dom}(\frac{1}{s}) = \emptyset$. Hence, $\text{Dom}(f) \cap V \neq \emptyset$, $\text{Dom}(g) \cap V \neq \emptyset$, but $\text{Dom}(f \cdot g) \cap V = \emptyset$. In particular, if $\mathcal{O}_{X,V}$ was defined for the present X and V as in [Definition 2.5.1](#), then it would not be a ring.

On the other hand, if the condition $\text{Dom}(f) \cap V \neq \emptyset$ would be replaced by $\text{Dom}(f) \cap V$ is dense in V , then the above hypothetical definition would become a ring. On the other hand, it would not be local. Indeed, it would have two maximal ideals, generated by the residue classes of x and y , respectively.

Lemma 2.5.6. THE LOCAL RING IS LOCAL IN TOPOLOGY: *If X is a variety, $V \subseteq X$ is an irreducible closed subvariety, and $Y \subseteq X$ is an open subset such that $Y \cap V \neq \emptyset$, then the natural restriction map $\mathcal{O}_{X,V} \rightarrow \mathcal{O}_{Y,Y \cap V}$ is an isomorphism (to define this restriction map we use that restriction of functions yield $K(X) \cong K(Y)$).*

Proposition 2.5.7. LOCALIZATION IS LOCALLY INDEED A LOCALIZATION: *If X is an affine variety and V is an irreducible closed subvariety, then $\mathcal{O}_{X,V} = A(X)_{I(V)}$ as subrings of $K(X)$.*

Corollary 2.5.8. THE LOCAL RING IS INDEED A LOCAL RING: *If X is a variety, and V an irreducible subvariety, then $\mathcal{O}_{X,V}$ is a local ring (in the algebraic sense, that is, it has a unique maximal ideal), with its maximal ideal $m_{X,V}$ being:*

$$m_{X,V} = \{ f \in \mathcal{O}_{X,V} \mid f|_{V \cap \text{Dom}(f)} \equiv 0 \}.$$

Example 2.5.9. Putting together Lemma 2.5.6 and Proposition 2.5.7 as in the proof of Corollary 2.5.8 we can give a lot of examples.

For example, let $x \in \mathbb{P}^n$. Then x has an affine neighborhood isomorphic to $\mathbb{A}^n$, where by translation we may also assume that x is the origin. Hence,

$$\mathcal{O}_{X,x} \cong k[x_1, \dots, x_n]_{(x_1, \dots, x_n)} = \left\{ \frac{f}{g} \in k(x_1, \dots, x_n) \mid g(0, \dots, 0) \neq 0 \right\}.$$

2.6 BÉZOUT'S THEOREM

Review of material that was covered in “Algebraic curves”

Theorem 2.6.1. *If C and D are projective plane curves, then*

$$\sum_{x \in C \cap D} I(x, C \cap D) = (\deg C)(\deg D).$$

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Chapter 3

Why scheme theory?

Classical algebraic geometry does not give satisfactory answers to the following:

- (1) What is a variety over a non-algebraically closed field?
- (2) What is a variety over $\mathbb{Z}$?
- (3) What is the “limit” of variety? Say, one takes $V(x(x + ty))$ as $t \rightarrow 0$? Is it just $V(x)$ or is it $V(x^2)$ in some adequate sense?
- (4) What is an infinitesimal deformation of a variety, that is, what is a variety and a deformation in a tangent direction?
- (5) What is the fiber of the morphism $\mathbb{A}_{x,y}^2 \rightarrow \mathbb{A}_t^1$ given by $(x^2, y) \mapsto t$ at $t = 0$. Is it $V(xy)$ or rather $V(x^2y)$ in an adequate setting?
- (6) If varieties are locally just reduced finitely generated k -algebras, then is there a notion on varieties that specializes locally to just finitely generated modules over these k -algebras. Moreover, we would like this notion to form an *abelian category*, that is, we would want to be able to take kernels, cokernels, and in general to make homological algebra with them, including taking derived functors, such as $\text{Ext}^i(-, -)$ or $\text{Tor}^i(-, -)$.
- (7) Is there an intrinsic definition of the main objects of algebraic geometry, so not as subsets of $\mathbb{P}^n$ or $\mathbb{A}^n$. It would make the construction of products much more easier for example.

In short, scheme and sheaf theory answers all the above questions.

Remark 3.0.1. Scheme theory is not the last answer. Depending on what we want to do, there are even more abstract theories. For example derived algebraic geometry creates the more general notion of derived schemes that work more functorially than schemes for many operations. Or, over more intricate fields, i.e., over perfectoid fields, it turns out that one needs more intricate objects such as perfectoid spaces. However, to understand these more intricate notions it is highly suggested that one is a master of scheme theory. And, after all, for many researchers, the abstraction of scheme theory is by far enough.

Chapter 4

Sheaves

Read carefully Section II.1. of Hartshorne. It is important to know what a direct limit is.

Definition 4.0.1. A *directed set* is a pair $(S, \leq)$ of a set S , and a reflexive and transitive relation $\leq$ on S , such that for each $x, y \in S$ there exists a $z \in S$ such that $x \leq z$ and $y \leq z$.

Definition 4.0.2. A *direct system* (A_*, f_{**}) of abelian group over a directed set $(I, \leq)$ is the data of

- (1) an abelian group A_i for every $i \in I$, and
- (2) a homomorphism of abelian groups f_{ij} for every $i \leq j$ in I , such that
 - (i) $f_{ii} = \text{id}_{A_i}$, and
 - (ii) $f_{jl} \circ f_{ij} = f_{il}$ for every $i \leq j \leq l$ in I .

Remark 4.0.3. If $(I, \leq)$ is considered to be a category such that there is arrow $i \rightarrow j$ whenever $i \leq j$, then a direct system of abelian group over a directed set $(I, \leq)$ is just a functor from this category to the category of abelian groups.

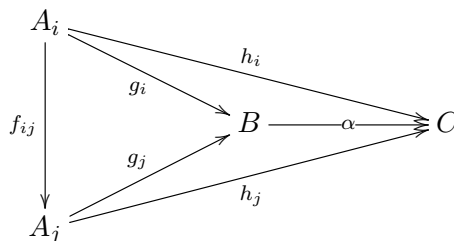
Definition 4.0.4. Let (A_*, f_{**}) be a direct system of abelian group over a directed set $(I, \leq)$. We say that (B, g_*) is the *direct limit* if

- (1) B is an abelian group, and
- (2) $g_i : A_i \rightarrow B$ are group homomorphisms for every $i \in I$ such that $g_j \circ f_{ij} = g_i$ for every $i \leq j$,

and additionally whenever we are given

- (3) an abelian group C , and
- (4) group homomorphisms $h_i : A_i \rightarrow C$ for every $i \in I$ such that $h_j \circ f_{ij} = h_i$ for every $i \leq j$,

then there is a unique homomorphism $\alpha : B \rightarrow C$ such that $\alpha \circ g_i = h_i$ for every $i \in I$. To summarize, we have the following diagram commutes for every $i \leq j$:



The notation is

$$\varinjlim_i A_i,$$

which is an abuse of notation, as the f_{ij} are also part of the data.

Proposition 4.0.5. *In the situation of Definition 4.0.4 the direct limit exists and can be described the following two ways:*

- (1) Take $\bigoplus_{i \in I} A_i$ and quotient out the subgroup generated by the elements $x - f_{ij}(x) \in \bigoplus_{i \in I} A_i$ for every $x \in A_i$ and every $i \leq j$. The homomorphisms $g_i : A_i \rightarrow \varinjlim_i A_i$ is induced by the structure homomorphisms $A_i \rightarrow \bigoplus_{i \in I} A_i$ of direct sums.
- (2) Take the disjoint union $\bigsqcup_{i \in I} A_i$ and quotient out the equivalence relation generated by letting x and $f_{ij}(x) \in \bigsqcup_{i \in I} A_i$ be equivalent for every $x \in A_i$ and every $i \leq j$. The homomorphisms $g_i : A_i \rightarrow \varinjlim_i A_i$ is induced by the structure maps $A_i \rightarrow \bigsqcup_{i \in I} A_i$.

Proof. Homework □

Here we also add some important examples to the material presented in Hartshorne.

First, two examples of pre-sheaves that are not sheaves.

Example 4.0.6. Consider the following functor $\mathcal{F}$ on the discrete topological space with two elements, say on $\{x, y\}$.

$$\begin{array}{ccccc}
 & \mathcal{F}(\{x, y\}) = \mathbb{Z} & & & \\
 & \swarrow \text{id}_{\mathbb{Z}} & \downarrow 0 & \searrow \text{id}_{\mathbb{Z}} & \\
 \mathcal{F}(\{x\}) = \mathbb{Z} & & & & \mathcal{F}(\{y\}) = \mathbb{Z} \\
 & \searrow 0 & \downarrow 0 & \swarrow 0 & \\
 & \mathcal{F}(\emptyset) = 0 & & &
 \end{array} \tag{4.0.a}$$

One can see that diagram (4.0.a) commutes, which is equivalent to say that $\mathcal{F}$ is a presheaf. *However, it is not a sheaf.* Indeed, choose $0 \in \mathbb{Z} = \mathcal{F}(\{x\})$, and $1 \in \mathbb{Z} = \mathcal{F}(\{y\})$. As $\{x\} \cap \{y\} = \emptyset$, by the gluing part of the sheaf axioms there should be a global sections $s \in \mathcal{F}(\{x, y\}) = \mathbb{Z}$ restricting to 0 on $\{x\}$ and to 1 on $\{y\}$. However, this is not possible by (4.0.a).

Example 4.0.7. Consider the functor $\mathcal{C}^{0,\infty}$ on the open sets of $\mathbb{R}$ that assigns to each open set $U \subseteq \mathbb{R}$ the continuous bounded functions on U , with restriction homomorphisms being just restrictions of functions. As restrictions of functions is the same if we do it through multiple restrictions or just a single one, this is a presheaf. *However, it is not a sheaf.* Indeed, if we take $\text{id}_{(n, n+2)} \in \mathcal{C}^{0,\infty}((n, n+2))$ for every $n \in \mathbb{Z}$, then this give a collection of elements on an open cover that glue, but they do not give a global section in $\mathcal{C}^{0,\infty}(\mathbb{R})$, as $\text{id}_{\mathbb{R}}$ is not bounded.

Second, two examples of surjective sheaf-homomorphisms that are not surjective when restricted to certain open sets.

Example 4.0.8. Let $x, y \in \mathbb{A}^1$ be two points, where $\mathbb{A}^1$ is the affine line as introduced in classical algebraic geometry. Set $\iota : Z = \{x, y\} \hookrightarrow \mathbb{Z}^1$. As $\mathbb{Z}_{\mathbb{A}^1}$ takes value $\mathbb{Z}$ on every non-empty open set, for any open set $\emptyset \neq U \subseteq \{x, y\}$ we have

$$\iota^{-1, \text{pre}} \mathbb{Z}_{\mathbb{A}^1}(U) = \varinjlim_{\iota(U) \subseteq V} \mathbb{Z}_{\mathbb{A}^1}(V) = \varinjlim_{\iota(U) \subseteq V} \mathbb{Z} = \mathbb{Z}$$

all maps of this direct system are $\text{id}_{\mathbb{Z}}$, and hence all elements are equivalent to an element of $\mathbb{Z}_{\mathbb{A}^1}(\mathbb{A}^1) = \mathbb{Z}$

Moreover, analyzing the above argument one can see that all maps of $\iota^{-1, \text{pre}} \mathbb{Z}_{\mathbb{A}^1}$ are identities. Hence $\iota^{-1, \text{pre}} \mathbb{Z}_{\mathbb{A}^1}$ is in fact the pre-sheaf of **Example 4.0.6**. In particular this is an example of when the sheafification is necessary to obtain $\iota^{-1} \mathbb{Z}_{\mathbb{A}^1}$, or with other words $\iota^{-1} \mathbb{Z}_{\mathbb{A}^1} \neq \iota^{-1, \text{pre}} \mathbb{Z}_{\mathbb{A}^1}$.

As $\iota^{-1, \text{pre}}$ agrees with $\mathbb{Z}_Z$ on a basis given by $\{x\}$ and $\{y\}$, by the sheaf axioms $\iota^{-1} = \mathbb{Z}_Z$. Hence, the sheaf homomorphism $\mathbb{Z}_{\mathbb{A}^1} \rightarrow \iota_* \iota^{-1} \mathbb{Z}_{\mathbb{A}^1}$ is not surjective on global sections. Indeed when evaluating the source and the target on $\mathbb{A}^1$ we obtain

$$\mathbb{Z}_{\mathbb{A}^1}(\mathbb{A}^1) = \mathbb{Z}, \text{ and } (\iota_* \iota^{-1} \mathbb{Z}_{\mathbb{A}^1})(\mathbb{A}^1) = \iota^{-1} \mathbb{Z}_{\mathbb{A}^1}(Z) = \mathbb{Z}^{\oplus 2}$$

by the definition of ι_*

by the above discussion

As there is no surjective group homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}^{\oplus 2}$ we obtain our non-surjectivity claim. On the other hand, $\mathbb{Z}_{\mathbb{A}^1} \rightarrow \iota_* \iota^{-1} \mathbb{Z}_{\mathbb{A}^1}$ is a surjective sheaf-homomorphism as can be checked for example on stalks.

Example 4.0.9. Consider $X = \mathbb{C}$ with the usual topology, which is sometimes called analytic and sometimes Euclidean (so not with the Zariski topology). Let $\mathcal{O}_{\mathbb{C}}^{\text{hol}}$ be the sheaf of holomorphic functions of one variable, and let $\mathcal{O}_{\mathbb{C}}^{\text{hol}, \times}$ be the sheaf on non-zero holomorphic functions. By adequately translating what we have learned in complex analysis we obtain that the following sequence of sheaves is exact:

$$0 \longrightarrow \mathbb{Z}_{\mathbb{C}} \xrightarrow{n \mapsto 2\pi i \cdot n} \mathcal{O}_{\mathbb{C}}^{\text{hol}} \xrightarrow{f \mapsto \exp(f)} \mathcal{O}_{\mathbb{C}}^{\text{hol}, \times} \longrightarrow 0$$

On the other hand, we also know from complex analysis that this sequence is not exact when evaluated on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Indeed, in that case if $\text{id}_{\mathbb{C}^*} \in \mathcal{O}_{\mathbb{C}}^{\text{hol}, \times}(\mathbb{C}^*)$ was in the image, then there would be a global logarithm function over $\mathbb{C}^*$.

Chapter 5

Localization

Scheme theory can be thought about the generalization of classical algebraic geometry to a setting where we replace “reduced finitely generated k -algebras” in [Theorem 2.1.12](#) by “arbitrary rings”. In particular, it can be thought as the marriage of classical algebraic geometry and commutative algebra. For that reason, it is particularly important to not just refresh the background in classical algebraic geometry ([Chapter 2](#)), but to also recall the commutative algebra background on a working level. That is, it should be recalled on a level on which we can effectively compute examples, and do computations and proofs of smaller statements in general quickly.

For understanding the nuances of the definition of schemes the most important is to recall *localization*. We suggest that you recall what we have learned about localization in “Rings and modules”. The most of it is contained in Sections 7.2 and 7.3 of the “Rings and modules” course notes.

It is particularly important to recall the following:

- (1) Localization by one element $f \in R$, which yields R_f , see Example 7.2.4.(3) in the “Rings and modules” notes.
- (2) Localization at a prime ideal $p \subseteq R$, which yields R_p , see Example 7.2.4.(4) in the “Rings and modules” notes.
- (3) Localization R_p of a ring that is an extension $S \subseteq R$ of another ring S at the prime ideal $p \subseteq S$, see Example 7.2.4.(6) in the “Rings and modules” notes. Here it is important to note that this is the “same” as localizing R as a module over S , where “same” means that both contains equivalence classes of fractions $\frac{r}{s}$ where $r \in R$ and $s \in S \setminus p$. Additionally two such fractions $\frac{r}{s}$ and $\frac{r'}{s'}$ are identified for both localizations if and only if there is a $u \in S \setminus p$ such that $u(rs' - r's) = 0$ as an element of R . The only difference is whether we remember the S -module structure of R_p or if we remember the ring structure on R_p .

Here it is also important to note that R_p is automatically a ring extension of S_p (the natural map S_p to R_p is injective!), and hence it is also a module over S_p .

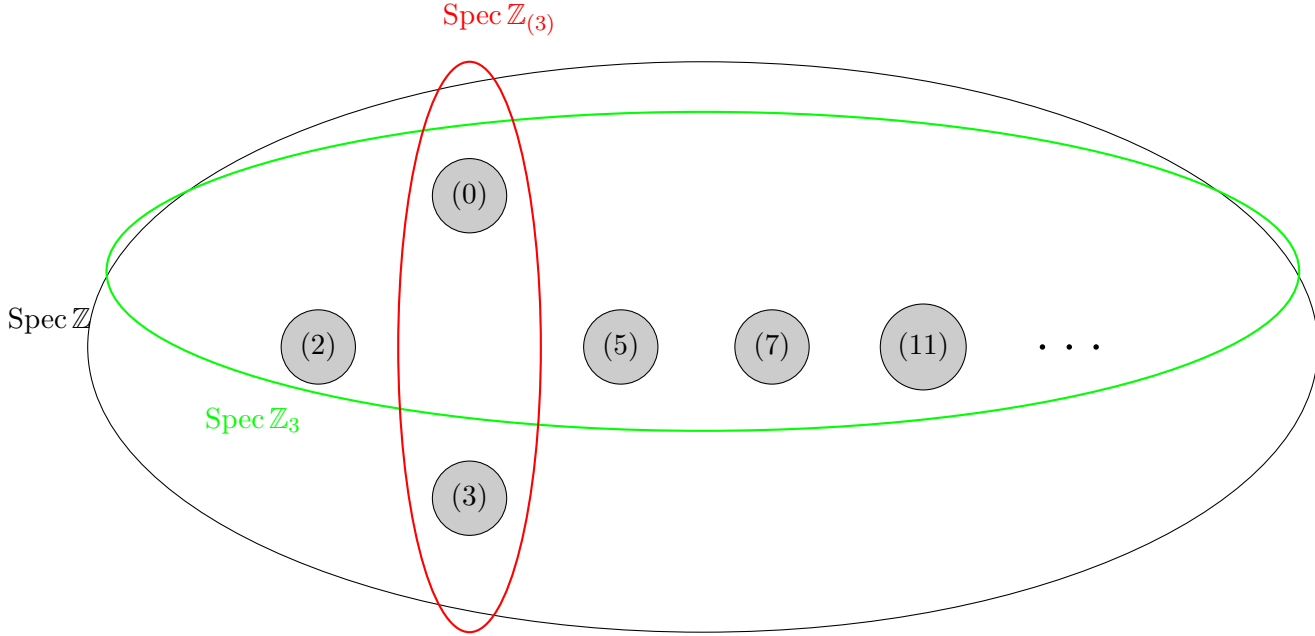
- (4) Proposition 7.3.9 of “Rings and modules” tells us how ideals behave when they are extended and contracted via the localization map. For example, point (iv) of Proposition 7.3.9 tells us that if we localize at a prime ideal $p \subseteq R$, the prime ideals of R_p will be of the form q^e for prime ideals $q \subseteq p \subseteq R$. This is also denoted many times by qR_p , which is a minor abuse of notation, as the map $R \rightarrow R_p$ is not always injective.

For example $k[x]_{(x)}$ has only two prime ideals $(x)^e$ and $(0)^e = (0)$. With other words localization restricts our attention to a subset of the *spectrum* of

$$\operatorname{Spec} R = \{ p \subseteq R \mid \}$$

(The spectrum will be endowed with a topology in Chapter 6, but for now let us think about it simply as a set.)

The following picture is a visualization of $\text{Spec } \mathbb{Z}$ and of how $\text{Spec } \mathbb{Z}_{(3)}$ and $\text{Spec } \mathbb{Z}_3$ should be thought of as a subset



The above pictures shows the intuitive idea that $\mathbb{Z}_{(3)}$ is much closer to $\mathbb{Q}$, while $\mathbb{Z}_3$ is much closer to $\mathbb{Z}$.

Indeed, $\mathbb{Z}_{(3)}$ is the best to understand by the fact that we invert in it anything BUT 3. Formally:

$$\mathbb{Z}_{(3)} \cong \left\{ \frac{a}{b} \in \mathbb{Q} \mid 3 \nmid b \right\}$$

On the other hand $\mathbb{Z}_3$ is the best to understand by the fact that we invert ONLY 3. Formally:

$$\mathbb{Z}_3 \cong \mathbb{Z} \left[\frac{1}{3} \right].$$

- (5) If R is a domain then every localization is contained in $\text{Frac } R$.
- (6) *Localization of a localization is a localization.* More precisely, iterated localizations fit into a commutative diagram, by the universal property of localization. That is, let $S \supseteq T$ multiplicative subsets of R , and denote by $S_{T^{-1}}$ the multiplicative subset

$$\left\{ \frac{s}{t} \in T^{-1}R \mid s \in S, t \in T \right\}.$$

Then we have a commutative diagram where the non-isomorphic arrows are the corre-

sponding structures homomorphisms of localizations:

$$\begin{array}{ccc}
 R & \xrightarrow{\alpha} & T^{-1}R \\
 \downarrow & \searrow \beta & \downarrow \gamma \\
 S^{-1}R & \xrightarrow{\cong} & (S_{T^{-1}})^{-1}(T^{-1}R) \\
 \Psi & & \Psi \\
 \frac{r}{s} & \longmapsto & \left(\frac{s}{1}\right)^{-1} \frac{r}{1} \\
 & & \\
 \frac{rt}{st'} & \longleftarrow & \left(\frac{s}{t}\right)^{-1} \frac{r}{t'}
 \end{array}$$

Here β is given by the universal property of the localization $R \rightarrow T^{-1}R$. Then, the isomorphism in one direction is given by the universal property of the localization $R \rightarrow S^{-1}R$ applied to $\gamma \circ \alpha$, and in the other direction by the universal property of the localization $T^{-1}R \rightarrow (S_{T^{-1}})^{-1}(T^{-1}R)$ applied to β .

Note that the above compatibility is evident when R is a domain, as in that case everything happens in $\text{Frac } R$, and hence the above two isomorphic rings are in fact the same subrings of $\text{Frac } R$. However, when R is not a domain (which is one of the main interesting case for scheme theory), then one has to exhibit the above standalone isomorphism.

Additionally, it is extremely useful to look at Exercise 4 of the 13th exercise sheet of “Rings and modules” this contains an algorithm of computing localization of rings of the types R/I . From that we learn the following, where $\xi : R \rightarrow R/I$ is the quotient homomorphism.

- (7) First, a remark about which multiplicative set one can use for such localizations, which is in fact loosely phrased in Exercise 4 of the 13th exercise sheet. If we have a multiplicatively closed set $S \subseteq R$, then $\xi(S)$ will be multiplicatively closed in R/I . Then, localizing R/I as an R -module at S is the “same” as localizing R/I as a ring at $\xi(S)$. We mean this in the sense of points (3) and (6) above. We let you make precise sense of it.
- (8) Similarly to what we discussed in (4), $\text{Spec } (R/I)$ is a subset of $\text{Spec } R$. For that reason, by abuse notation the appearance of a prime ideal $p \subseteq R$ in both of them is denoted by p , despite of the fact that in $\text{Spec } R/I$ we should really call it $\xi(p)$ (note that these are the prime ideals for which $p \supseteq I$).

Combining the previous paragraph and the previous point, for p as above, we denote the localization of R/I at p (that is really at $\xi(p)$) by $(R/I)_p$.

- (9) In Exercise 4 of the 13th exercise sheet of “Rings and modules” it is shown that if $S \subseteq R$ is a multiplicatively closed set, then $S^{-1}(R/I) \cong S^{-1}R/I^e$. Note that if I is given by generators, then I^e is generated by the images of these generators in the localization.

In particular, for $p \in \text{Spec } R$, and for a primary decomposition $I = \bigcap_{i=1}^s I_i$ (recall what primary decomposition is from Section 8.5 of the “Rings and modules” notes):

- (i) if $p \notin \text{Spec } (R/I)$, then $(R \setminus p)^{-1}(R/I) = 0$, and
- (ii) if $p \in \text{Spec } (R/I)$, then

$$(R/I)_p \cong R_p/I^e = R_p/\bigcap_{p \supseteq I_i} I_i^e$$

Let’s look at an example of the machinery explained in point (9) above

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Example 5.0.1. Consider $I = (x^2y, xy^2) \subseteq k[x, y]$ for any field k . A minimal primary decomposition of I is given by

$$I = (x) \cap (y) \cap \underbrace{(x^3, x^2y, xy^2, y^3)}_{\substack{\uparrow \\ = (x, y)^3 = J}}$$

As J is (x, y) -primary, $\text{Spec} \left(k[x, y]/I \right)$ contains of those prime ideals that either contain (x) or they contain (y) . If we localize outside of this set, then we obtain 0. Since x and y play symmetric roles in this example, there are two types of prime ideals that contain (x) and (y) :

- (1) If a prime ideal $p \subseteq k[x, y]$ contains both (x) and (y) , then it has to contain (x, y) . As the latter is a maximal ideal, in this case $p = (x, y)$. This unfortunately contains all the ideals in the primary decomposition, so in this case we have

$$\left(k[x, y]/I \right)_{(p)} = \underbrace{A/(x)_A \cap (y)_A \cap (x^3, x^2y, xy^2, y^3)_A}_{\substack{\uparrow \\ A = k[x, y]_{((x, y))}}} = A/(x^2y, xy^2)_A$$

where subindex A means ideal generated in A .

- (2) If a prime ideal $p \subseteq k[x, y]$ contains (x) but not y , then in particular it also does not contain J (as J is (x, y) -primary). Hence, we obtain that

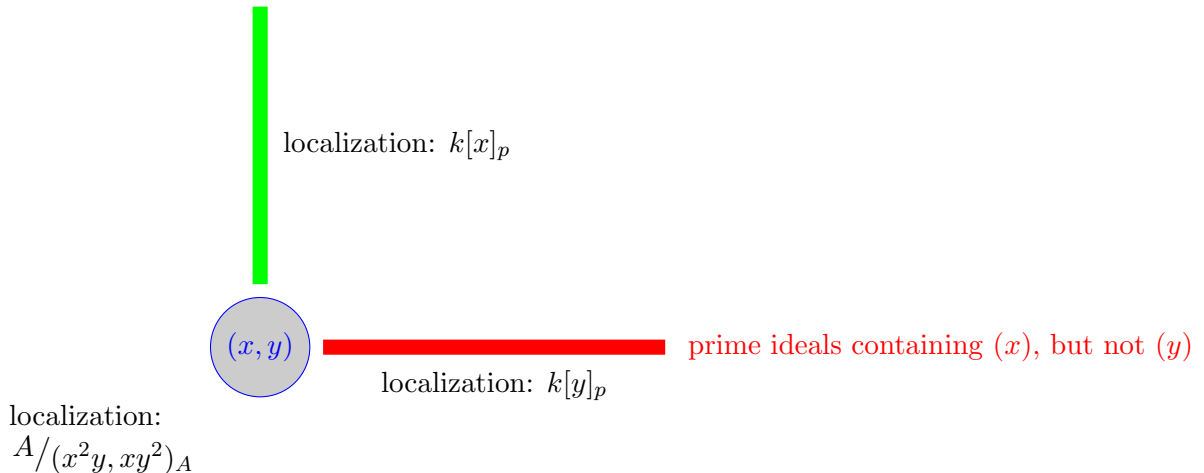
$$\left(k[x, y]/I \right)_{(p)} = \underbrace{B/Bx}_{\substack{\uparrow \\ B = k[x, y]_p}} \uparrow \left(k[x, y]/x \right)_p = \underbrace{k[y]_p}_{\substack{\uparrow \\ p \text{ is regarded as an ideal of } k[y] \text{ via the identification } k[x, y]/(x) \cong k[y]}}$$

applying the relation $\left(\frac{R}{L} \right)_p \cong R_p/L^e$ in the backwards direction for $L = (x)$

So, for example if $p = (x)$, then it corresponds to the prime ideal $(0) \subseteq k[y]$, and hence we obtain $\left(k[x, y]/I \right)_{(p)} \cong k(y)$. If $p = (x, y - 1)$, then it corresponds to the prime ideal $(y - 1) \subseteq k[y]$, and hence we obtain $\left(k[x, y]/I \right)_{(p)} \cong k[y]_{(y-1)}$.

We summarize what we showed in the following picture:

prime ideals containing (y) , but not (x)



By a similar argument we also obtain that

$$\left(k[x, y]/I\right)_y = k[x, y]_y/(x) = \left(k[x, y]/(x)\right)_y = k[y]_y = k[y, y^{-1}] \quad (5.0.a)$$

where we similarly to the case of the localization by a prime ideal, in some expressions here y is treated as the element of $k[x, y]$, and in some as the image of this element in $k[x, y]/I$, and in some as an element of $k[y]$.

We note that the non-maximal components of the primary decomposition of I are called the *embedded components* of R/I , which will have geometric meaning in scheme theory.

Chapter 6

Schemes and affine schemes

From now on in the entire course, k is an arbitrary field (so not necessary algebraically closed as it was in [Chapter 2](#)). We will always indicate when k is algebraically closed.

In this section we cover the part of Section II.2 of Hartshorne until the top of page 76, that is until the sentence “Next we will define and important..”, as well as Prop 2.6 of Section II.2. of Hartshorne. With other words we cover everything from Section II.2. of Hartshorne, except pages 76 and 77. Those we will cover in ??

We follow the book most of the time, and we only note a few important things :

Remark 6.0.1. In the definition of $\text{Spec } A$, it is worth mentioning that $V(I)$ actually inherits a subspace topology. Its closed subsets are intersection of closed subsets of $\text{Spec } A$ with $V(I)$, that is, sets of the form $V(I) \cap V(J)$. However, then these are exactly the sets of the form $V(I + J)$, and hence by the correspondence theorem we obtain that $V(I)$ with this subspace topology is topologically isomorphic to $\text{Spec } (A/I)$.

We also note that the above method puts the same topological space structure of $V(I)$ and on $V(\sqrt{I})$, which is then equivalently the topological space of $\text{Spec } (A/I)$ and of $\text{Spec } (A/\sqrt{I})$ as well. However, the scheme structure of $\text{Spec } (A/I)$ and of $\text{Spec } (A/\sqrt{I})$ is in general different. This is an important point to which we will get back later.

Remark 6.0.2. The definition of $\mathcal{O}_{\text{Spec } A}$ is secretly the sheafification of a much more intuitive pre-sheaf, and knowing this helps to streamline some of the arguments. This pre-sheaf is

$$\begin{array}{ccc} \text{Spec } A \supseteq U & \mapsto & \mathcal{O}_{\text{Spec } A}^{\text{pre}}(U) = S(U)^{-1}A \\ \uparrow & & \\ \boxed{\text{open}} & & \end{array}$$

where

$$S(U) = \{ f \in A \mid \forall p \in U: f \notin p \}$$

The reason why we have $(\mathcal{O}_{\text{Spec } A}^{\text{pre}})^+ = \mathcal{O}_{\text{Spec } A}$ is that for any $p \in A$ we have

$$\varinjlim_{p \in U} (S(U)^{-1}A) \xrightarrow[\cong]{\phi} A_p$$

where ϕ is given by the universal property of direct limit applied to the ring homomorphisms

$$S(U)^{-1}A \rightarrow A_p$$

given by iterated localization (see [Chapter 5](#)). We show that ϕ is surjective and injective:

ϕ is surjective: take $\frac{a}{t} \in A_p$, then

$$\begin{array}{ccc} \frac{a}{t} & = & \phi\left(\frac{a}{t}\right) \\ \uparrow & & \uparrow \\ \in A_p & & \in S(D(t))^{-1}A \end{array}$$

ϕ is injective: Suppose $\phi\left(\frac{a}{t}\right) = 0$ for $\frac{a}{t} \in S(U)^{-1}A$. That is, there exists $s \in A \setminus p$ such that $sa = 0$. However, then $\frac{a}{t}$ is already zero in $U \cap D(s)$, and hence $\frac{a}{t}$ yields the zero element of the inverse limit.

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Remark 6.0.3. Prop 2.2 of Hartshorne Section II.2 is not quite sufficiently stated for how it is applied. It is important to add to point (a) that the isomorphism is via the isomorphism of **Remark 6.0.2** or equivalently by the evaluation isomorphism of the sections of $\mathcal{O}_{\text{Spec } A}$ at p (the values of the sections of $\mathcal{O}_{\text{Spec } A}$ according to Hartshorne's definition at $p \in \text{Spec } A$ are in A_p).

For point (b) it again it is important to remember that the isomorphism is given either by the natural homomorphism (given by sheafification):

$$A_f = \mathcal{O}_{\text{Spec } A}^{\text{pre}}(D(f)) \rightarrow \left(\mathcal{O}_{\text{Spec } A}^{\text{pre}}\right)^+(D(f)) = \mathcal{O}_{\text{Spec } A}(D(f))$$

or equivalently by sending $\frac{a}{f^n} \in A_f$ to the section that evaluates at $p \in A_p$ to $\frac{a}{f^n} \in A_p$.

It is also curcial to add the following examples to the discussion of Section II.2 of Hartshorne.

Example 6.0.4. Revisit **Example 5.0.1** from a scheme theoretic point of view: generic points, stalks, embedded components, etc.

Conclude from (5.0.a), that $D(x) \cong D(y) \cong \mathbb{A}^1 \setminus \{0\}$.

Beef up the example to the case of

$$I = (xy^3, x^2y^2) = (x) \cap (y^2) \cap (x, y)^4.$$

In this case, $D(x)$ is an infinitesimal fattening of $\mathbb{A}^1 \setminus \{0\}$.

Example 6.0.5. There will be two exercises on the exercise sheet (Exc 2.10 and 2.11) asking you to show that points of schemes over non-closed fields correspond to Galois orbits of schemes over algebraically closed fields.

Example 6.0.6. Spectrum of a field.

Example 6.0.7. Spectrum of a DVR, say $\text{Spec } k[x]_{(x)}$.

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Example 6.0.8. $\text{Spec } \mathbb{Z}$.

Example 6.0.9. Let R be a ring and $p \subseteq R$ a prime ideal. Then we have natural homomorphisms of rings

$$R \longrightarrow R_p \longrightarrow R_p/pR_p \cong \text{Frac}(R/p)$$

the natural homomorphism $R \rightarrow R_p$ induces this isomorphism by Additionally, $S^{-1}R$ is a flat R -modules, exercise 2.(d) of Sheet 11 of "Rings and modules", see **Remark 6.0.10** for additional commutative algebra review essential around this topic

Hence, if $X = \text{Spec } R$, then we obtain morphisms of schemes

$$\text{Spec } \text{Frac}(R/p) = \text{Spec } k(p) \longrightarrow \text{Spec } R_p = \text{Spec } \mathcal{O}_{X,p} \longrightarrow \text{Spec } R = X$$

These are particularly important and frequently used morphisms.

If R is a domain, and $p = (0)$, then p is called the generic point and it is typically denoted by η . In general the generic points of the schemes are the minimal primes of (0) (see **Section 7.3**) for a discussion on minimal primes.

Remark 6.0.10. We recall additional facts about localization around the isomorphism $R_p/pR_p \cong \text{Frac}(R/p)$:

- (1) It is very important to recall the localization of a module M over a ring R along a multiplicatively closed set $S^{-1} \subseteq R$. This yields an $S^{-1}R$ -modules $S^{-1}M$, see Def 7.2.5 of the “Rings and modules” notes.
- (2) $S^{-1}M \cong M \otimes_R S^{-1}R$, by Exercise 1 of Sheet 11 of “Rings and modules”.
- (3) $S^{-1}R$ is a flat R -modules, exercise 2.(c) of Sheet 11 of “Rings and modules”
- (4) For every ideal $I \subseteq R$ there is a natural evaluation map $I \otimes_R S^{-1}R \rightarrow I^e \subseteq S^{-1}R$, which is surjective by the definition of I^e . By the flatness of $S^{-1}R$ over R additionally this evaluation map is injective (as $(-) \otimes_R S^{-1}R$ takes $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$ to an exact sequence). Hence, $I \otimes_R S^{-1}R \rightarrow I^e$ is an isomorphism, which is natural with respect to inclusion of ideals.
- (5) The above point is important to prove $R_p/pR_p \cong \text{Frac}(R/p)$. Indeed, here pR_p is just another notation for p^e , and then one applies $(-) \otimes_R R_p$ to the exact sequence

$$0 \longrightarrow p \longrightarrow R \longrightarrow R/p \longrightarrow 0$$

This yields that $R_p/pR_p \cong (R/p) \otimes_R R_p$. Then, one just has to identify the latter with $\text{Frac}(R/p)$. This is done by using point (7) of Chapter 5.

- (6) Back to the general situation (so not the specific situation of the previous point), if N is another R -module, then $M \otimes_R (N \cdot \text{Ann}(M)) = 0$. Indeed, for this, it is enough to show that the simple tensors in this tensor product are zero (as they generate). However, if $m \in M$, $n \in N$ and $r \in \text{Ann}(M)$, then

$$m \otimes (n \cdot r) = rm \otimes n = 0 \otimes n = 0$$

- (7) We have $M \otimes_R N \cong M \otimes_R \left(N / N \cdot \text{Ann}(M) \right)$ via the natural homomorphisms on the two coordinates. Indeed, one applies $(-) \otimes_R M$ to the exact sequence

$$0 \longrightarrow N \cdot \text{Ann}(M) \longrightarrow N \longrightarrow N / N \cdot \text{Ann}(M) \longrightarrow 0$$

and uses the previous point.

- (8) As $\text{Ann}(M)$ acts trivially on both coordinates in $M \otimes_R \left(N / N \cdot \text{Ann}(M) \right)$, we can even replace R by $R/\text{Ann}(M)$. That is, we have

$$M \otimes_R N \cong M \otimes_{(R/\text{Ann}(M))} \left(N / N \cdot \text{Ann}(N) \right) \quad (6.0.a)$$

.

Using the above statements, for example one can prove that

$$\left(I / I^2 \right) \otimes_R S^{-1}R \cong I^e / (I^e)^2 \quad (6.0.b)$$

To prove (6.0.b), take the following exact sequence of R -modules

$$0 \longrightarrow I^2 \longrightarrow I \longrightarrow I / I^2 \longrightarrow 0$$

We apply $(-) \otimes_R S^{-1}R$, and we use that $(I^e)^2 = (I^2)^e$ by definition. This yields the following exact sequence, which concludes the proof of (6.0.b)

$$0 \longrightarrow I^2 \otimes_R S^{-1}R = (I^2)^e = (I^e)^2 \longrightarrow I \otimes_R S^{-1}R = I^e \longrightarrow (I/I^2) \otimes_R S^{-1}R \longrightarrow 0$$

In the case when $I = m$ is maximal ideal and $S = R \setminus m$ (or equivalently we consider localization at m), then m is in the annihilator of the R -modules m/m^2 . Hence, we have

$$(m/m^2) \otimes_R R_m \cong \underset{\substack{\uparrow \\ (6.0.a),}}{m/m^2} \otimes (R/m) \left(\frac{R_m}{mR_m} \right) \cong \underset{\substack{\uparrow \\ R/m \cong R_m/mR_m}}{m/m^2} \quad (6.0.c)$$

That is, putting together (6.0.b) and (6.0.c), we obtain that $(m/m^2) \cong m^e / (m^e)^2$. With other words the Zariski tangent spaces of $m \in \text{Spec } A$ can be computed two different ways if $n \subseteq A$ is a maximal ideal, see exercise II.2.8 of Harshorne. The argument is as follows.

6.1 CLASSICAL VARIETIES AS SCHEMES

Finally we remark that Prop 2.6 of Section II.2. of Hartshorne moves our entire classical theory of varieties into scheme theory. From now we identify the two point of views of classical algebraic geometry. That is, when we talk about a classical variety over an algebraically closed field, we think about the classical object and the corresponding scheme at the same time. So in particular:

Remark 6.1.1. Some important take home messages from the proof of Prop 2.6 of Section II.2. of Hartshorne:

- (1) We identify a classical affine variety X with coordinate ring $A(X)$ with the scheme $\text{Spec } A(X)$.
- (2) In scheme theory classical varieties obtain extra points, that is, one point for each closed irreducible subspace. For example for $\mathbb{A}_k^1$ over an algebraically closed k (see Definition 6.1.2), there are the usual points that we are used to in the classical theory corresponding to $(x - c) \subseteq k[x]$ (also called the closed points), and there is one extra point η corresponding to $(0) \subseteq k[x]$ called the generic point.
- (3) By Example 6.0.9 we obtain many useful non-classical schemes associated to a classical variety. These are frequently used.

SUGGESTION: for all statements about scheme theory, it is highly suggested that you start understanding them by thinking through what they mean for classical varieties.

Definition 6.1.2. If R is a ring, then $\mathbb{A}_R^n = \text{Spec } R[x_1, \dots, x_n]$.

Note that the above definition is compatible with classical algebraic geometry. That is, the scheme corresponding to $\mathbb{A}_k^n$ is indeed $\text{Spec } k[x_1, \dots, x_n]$ as it was shown in Prop 2.6 of Section II.2. of Hartshorne.

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08.10.2020.

Chapter 7

Dimension theory of rings

We recall that until the end of the course, k is an arbitrary field (so not necessarily algebraically closed as it was in [Chapter 2](#)). We will always indicate when k is algebraically closed.

We stop for a little bit with the discussion of Section II.2 of Hartshorne, and we review and complement our commutative algebra knowledge about dimension theory.

The dimension of a scheme is an invariant associated to its underlying topological space, using notion of dimension associated to a topological space:

7.1 DIMENSION OF A TOPOLOGICAL SPACE

Let us consider a closed subspace $V(I)$ of $\text{Spec } A$ for some ring A . As $V(I) = V(\sqrt{I})$, let us assume that I is radical. We have that $V(I)$ is irreducible if and only if I is prime. To see this equivalence, in one direction take $a, b \in R \setminus I$ such that $ab \in I$. As every radical ideal is the intersection of the prime ideal containing it, $V(I) \setminus V(a) \neq \emptyset$ and $V(I) \setminus V(b) \neq \emptyset$. That is, $V(a) \cap V(I)$ and $V(b) \cap V(I)$ are proper closed subsets of $V(I)$. Additionally, their union is $V(I)$ by the assumption $ab \in I$. Hence $V(I)$ is not irreducible. And, in the other direction one basically just reverses this argument.

Definition 7.1.1. If X is a topological space, then its dimension has values in $\mathbb{N} \cup \{\infty\}$ and it is

$$\dim X := \sup \{ n \mid \exists Z_0 \subsetneq Z_1 \cdots \subsetneq Z_n \text{ irreducible closed subsets of } X \}.$$

If $X = \text{Spec } A$, then $\dim X$ is the dimension as a topological space.

Remark 7.1.2. As chains of irreducible closed subsets of $\text{Spec } A$ are in one to one correspondence with chains of prime ideals of A (inclusion reverses!), we have that $\dim(\text{Spec } A) = \dim A$, where $\dim A$ is the Krull-dimension as we have learned it in “Rings and modules”

Remark 7.1.3. By [Definition 7.1.1](#), for $\text{Spec } A$ to have finite dimension, one needs that ascending and descending chains of prime ideals stabilize. This is related but not equivalent to A being Noetherian. More precisely:

- (1) It is true that most examples we know of Noetherian rings are of finite dimension. In some sense this is what we show in the present chapter. For example, a precise statement is that **local** Noetherian rings have finite dimension ([Corollary 7.4.7](#)).
- (2) There are Noetherian rings of infinite dimension. The first example was due to Nagata, and it is a localization of the polynomial ring in countably many infinite variables.

- (3) There are non-Noetherian rings of finite dimension. For example: $k[x_1, x_2, \dots] / (x_1^2, x_2^2, \dots)$ has a single prime ideal, which is also its nil-radical. Hence it has finite dimension, but it is not Noetherian.

Definition 7.1.4. If $p \subseteq R$ is a prime ideal in a ring, then its height $\text{ht } p$ is defined to be the supremum of all the integers n , for which there is a chain of prime ideals

$$p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_{n-1} \subsetneq p_n = p.$$

In terms of an affine scheme $X = \text{Spec } A$, $\text{ht } p$ is called the *codimension* of the closed set $V(p) \subseteq X$.

Example 7.1.5. As $\mathbb{Z}$ and $k[x]$ are PID's, we have $\dim \text{Spec } \mathbb{Z} = 1$ and $\dim k[x] = 1$. See Example 6.1.2.(2) of the “Rings and modules” notes.

This is luckily compatible with what we learned in “Algebraic curves”, as the classical $\mathbb{A}^1$ is by this point identified with $\text{Spec } k[x]$ for k algebraically closed.

However, it pertains to many non-classical schemes as well. For example $\dim \mathbb{A}_k^1 = 1$ also if k is not algebraically closed.

Another non-classical example is using the fact that the dimension is a topological notion. So, the nilpotent thickening $\text{Spec} \left(k[x, y] / (y^2) \right)$ of $\mathbb{A}_k^1$ also has dimension 1. Indeed, for this it is enough to see that the two topologies are isomorphic, for which in turn it is enough to show that the natural quotient homomorphism given by the following composition yields an isomorphism on topology

$$R = k[x, y] / (y^2) \rightarrow k[x, y] / (y^2) \cong k[x].$$

Indeed, if one passes to Spec then the above quotient homomorphism expresses $\mathbb{A}^1$ as the $V(\text{rad}(R))$ in $\text{Spec } R$. As $\text{rad}(R)$ is the radical of (0) , we see that $V(\text{rad}(R)) = \text{Spec } R$ topologically.

Example 7.1.6. As $\text{trdeg}_k k(x_1, \dots, x_n) = n$, and so $\dim \text{Spec } k[x_1, \dots, x_n] = n$. See Thm 6.1.11 and Example 6.1.2.(3) of the “Rings and modules” notes.

This is luckily compatible with what we learned in “Algebraic curves”, as the classical $\mathbb{A}^n$ is by this point identified with $\text{Spec } k[x_1, \dots, x_n]$ for k algebraically closed.

We also note that as $k[x_1, \dots, x_n]$ is a UFD, and x_i are irreducible, it is easy to see that (x_i) are prime ideals of height 1. However, the same is true for any irreducible $f \in k[x_1, \dots, x_n]$. That is, (f) is a prime ideal of height 1.

The next question is what is the dimension of the spectra of rings such as $k[x_1, \dots, x_n] / I$ or of $\mathbb{Z}[x_1, \dots, x_n]$ and of its quotients. This is addressed in [Section 7.2](#) and in [Section 7.3](#), together with the necessary commutative algebra background.

7.2 ADDITIVITY OF DIMENSION AND HEIGHT

In this section we are proving the following theorem.

Theorem 7.2.1. *Consider the following situation:*

- R is a finitely generated k -algebra and also a domain, and
- $p \subseteq R$ is a prime ideal.

Then $\dim R = \dim (R/p) + \text{ht } p$ [or equivalently $\dim \text{Spec } R = \dim V(p) + \text{codim } V(p)$, which is surprisingly not true for general schemes, see Example 7.2.2].

Before proving [Theorem 7.2.1](#), let us note in the following example that it does not hold if R is not a finitely generated k -algebra. That is, [Theorem 7.2.1](#) works in the classical setting and in its generalizations over non-closed fields.

Example 7.2.2. Consider the situation:

- $A = k[x]_{(x)}$,
- $R = A[y]$, and
- $p = (xy - 1)$.

Then:

- $R/p \cong k(x)$, and hence $\dim(R/p) = 0$.
- $\text{ht } p = 1$ by [Theorem 7.3.4](#).
- R is the localization of $k[x, y]$ at $S = k[x] \setminus (x)$. Hence, by the behavior of prime ideals with respect to localization, we know that the prime ideals of R correspond to prime ideals of $k[x, y]$ avoiding S . In particular, $\dim R \leq \dim k[x, y] = 2$, and we can exhibit a chain of prime ideals of length 2 in R :

$$(0) \subsetneq (y) \subsetneq (x, y)$$

Hence, we have $\dim R = 2$.

So, we obtained that

$$\dim R = 2 \neq 0 + 1 = \dim(R/p) + \text{ht } p$$

This shows that the finitely generated k -algebra assumption indeed is essential in [Theorem 7.2.1](#).

Note that instead of the above choices we could have chosen also $A = \mathbb{Z}_{(q)}$ for some prime $q \in \mathbb{Z}$ and $p = (py - 1)$. In that case we would have $R/p \cong \mathbb{Q}$.

Example 7.2.3. Let $X = \text{Spec} \left(k[x_1, \dots, x_n] / p \right)$ for some prime ideal $p \subseteq k[x_1, \dots, x_n]$. If k is algebraically closed, this means that X is the classical affine variety $V(p)$ in $\mathbb{A}^n$. However, our setting works also over non-closed k , and we have $\dim X = n - \text{ht } p$ by [Theorem 7.2.1](#).

Note, that a prime polynomial f in any ring R generates a prime ideal. If R is a UFD, as when $R = k[x_1, \dots, x_n]$, then one can show by hand that $\text{ht}(f) = 1$ (see [Example 7.1.6](#)). (If R is in general Noetherian, then the statement that $\text{ht}(f) = 1$ will be shown in [Theorem 7.3.4](#).)

So, by considering $V(f)$, any prime element in $f \in k[x_1, x_2]$ yields an example of an affine curve in $\mathbb{A}_k^2$, and any prime element in $f \in k[x_1, x_2, x_3]$ yields an example of an affine surface in $\mathbb{A}_k^3$. The question is how does one decide if f is a prime element. For this it is important to know the Gauss lemmas and the Eisenstein criterion (Section 2.7 of the “Anneaux et coprs” notes).

We can use for example the Eisenstein criterion for $k[x_1, \dots, x_n] = (k[x_1, \dots, x_{n-1}])[x_n]$. That is, if $f \in k[x_1, \dots, x_n]$, we may write it as $\sum_{i=0}^d f_i(x_1, \dots, x_{n-1})x_n^i$ such that $f_d \neq 0$, and then f is irreducible if there is an irreducible polynomial $g \in k[x_1, \dots, x_{n-1}]$, such that $g \nmid f_n$, $g \mid f_i$ for $i < d$ and $g^2 \nmid f_0$. For example, $x_2^2 - x_1(x_1 + 1)$ is irreducible, because $x_1 \nmid 1$, $x_1 \mid x_1(x_1 + 1)$ and $x_1^2 \nmid x_1(x_1 + 1)$.

Similarly, $x_1^2 + x_2^2 + x_3^2$ is irreducible if and only if $\text{char } k \neq 2$. Indeed, if $\text{char } k = 2$, then $x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2$, so it is not irreducible. So, it is enough to show using the Eisenstein criterion that in the other cases, $x_1^2 + x_2^2 \in k[x_1, x_2]$ is the product of two different irreducibles. Indeed, if $\text{char } k \neq 2$, then there are two different 2nd roots of unity in k , 1 and -1 , and there is a 2-nd root of -1 too (because k is algebraically closed), which we denote by ε . In particular, $x_1^2 + x_2^2 = (x_1 + \varepsilon x_2)(x_1 - \varepsilon x_2)$, showing the factorization into two different irreducibles. We

might also want to argue that the above linear polynomials are indeed irreducibles. This follows directly from the additive property of the degree: $\deg(gh) = (\deg f) + (\deg g)$ for any $f, g \in k[x_1, \dots, x_n]$ (Rem 1.4.26 of the “Anneaux et corps” notes). That is, a degree 1 polynomial cannot be written as a product of two positive degree polynomials.

The above irreducibility of the linear polynomials $x_1 + \varepsilon x_2$ and $x_1 - \varepsilon x_2$ can also be shown by using one of the Gauss lemmas, saying that a primitive element of $R[t]$ is irreducible if and only if it is irreducible as an element of $\text{Frac}(R)[t]$. Here primitive means that the greatest common divisor of the coefficients is 1. Here this means that we need to show that for example $x_1 + \varepsilon x_2$ is irreducible as an element of $k(x_1)[x_2]$. However, this is obvious because a linear polynomial of one variable is always irreducible over a field.

Now, we turn to the proof of **Theorem 7.2.1**. First, in one direction the inequality is straightforward:

Lemma 7.2.4. *If $p \subseteq R$ is a prime ideal, then $\dim(R/p) \leq \dim R - \text{ht } p$.*

Proof. Set $i = \dim(R/p)$ and $j = \text{ht } p$. We have to prove that $i + j \leq \dim R$. By the definition of height there is a chain $p_0 \subsetneq \dots \subsetneq p_j = p$. By the definition of dimension, $(0) = q_0 \subsetneq \dots \subsetneq q_i$ is a chain of prime ideals of R/p . By the correspondence theorem, this induces a chain $p = p_j \subsetneq \dots \subsetneq p_{i+j}$. If we put together these two chains, we obtain that $i + j \leq \dim R$. $\square$

So, to prove **Theorem 7.2.1**, we only have to prove the reverse equality (compared to **Lemma 7.2.4**). The idea is basically linear algebra, that is, we find a basis “adapted” to p , that is, the first $\text{ht } p$ elements are in p , and the other elements give a basis of R/p . The only trick is that basis here means *transcendence basis*, and hence as always the ideas with transcendence bases are the same as in linear algebra, but it takes much more time to make them precise.

To make the above idea precise one needs a Noether normalization type statement. This is **Theorem 7.2.5**. The proof is rather similar to the one we learned in “Rings and modules” for the Noether normalization, although the statement itself is fundamentally different. For example **Theorem 7.2.5** works only in the polynomial ring, unlike the actual Noether normalization. We refer to **Lemma 7.2.8** for an explanation how **Theorem 7.2.5** induces a transcendence basis of R/p .

Material very similar to what we learned in “Rings and modules”

Theorem 7.2.5. *In the polynomial ring $R = k[z_1, \dots, z_n]$ let p be a prime ideal of height 1. Then there are algebraically independent elements $y_1, \dots, y_n \in R$ such that*

- (1) $(y_1) = p$,
- (2) and R is integral over $S = k[y_1, \dots, y_n]$.

[Or equivalently, there is a finite dominant morphism $\text{Spec } k[z_1, \dots, z_n] \rightarrow \text{Spec } k[y_1, \dots, y_n]$, such that (the pullback of) y_1 is a defining equation of $V(p) \subseteq \text{Spec } k[z_1, \dots, z_n]$.]

Proof. Note that R is a UFD. As $\text{ht } p = 1$ we can find $0 \neq r \in p$. By the prime property then there will be an irreducible factor y_1 of r , such that $y_1 \in p$. As (y_1) is a non-zero prime ideal, by the assumption $\text{ht } p = 1$, we obtain that $p = (y_1)$. We can write then $y_1 = f(z_1, \dots, z_n)$ for some polynomial f in k and in n -variables.

Up to reordering the variables, we may assume that f is a non-constant polynomial in z_1 . Set $d := \deg f$, and let

$$C = \left\{ (c_1, \dots, c_n) \in \mathbb{N}^n \mid \sum_{i=1}^n c_i \leq d \right\}$$

Choose then an integer $N > 1$ such that:

- (1) for every $(a_1, \dots, a_n) \in C \setminus \{(d, 0, \dots, 0)\}$ we have

$$d < \sum_{i=1}^n a_i N^{i-1}$$

As all such sums are at least as big as N , this is true for $N \geq d + 1$.

- (2) for every $(a_1, \dots, a_n) \neq (b_1, \dots, b_n) \in C$ we have

$$\sum_{i=1}^n a_i N^{i-1} \neq \sum_{i=1}^n b_i N^{i-1} \quad (7.2.a)$$

This is doable as for each $(a_1, \dots, a_n) \in C$ the expression $\sum_{i=1}^n a_i N^{i-1}$ is a polynomial in N , and additionally for different choices of elements of C this polynomial is different. There is one special polynomial out of these that is the one associated to $(d, 0, \dots, 0)$, which is just the constant d polynomial. The other polynomials are all monotone increasing for large values of N . Additionally, as they are all different polynomials, no two of them have common values for big enough N . As C is finite, this means that for setting N big enough we can actually make (7.2.a) hold.

For $i = 2, \dots, n$, fix then $y_i := z_i - z_1^{N^{i-1}}$. We are left to show that R is integral over $S = k[y_1, \dots, y_n]$, for which it is enough to show that z_1 is integral over S (using that the integral elements over S form a subring of R).

Claim. *The polynomial*

$$f(z_1, y_2 + z_1^N, \dots, y_n + z_1^{N^{n-1}}) - y_1 \in (k[y_1, \dots, y_n])[z_1] \quad (7.2.b)$$

has leading term of the form cz_1^α with $c \in k \setminus \{0\}$.

Proof. If $f = cz_1^d$, then the leading term is cz_1^d . So suppose that $f \neq cz_1^d$. Then, by the choice of N , the leading term is of the form

$$\prod_{i=1}^n z_i^{a_i N^{i-1}}$$

for some $(a_1, \dots, a_n) \in C$ by our choice of N . □

So, the above claim shows that, z_1 is indeed integral over S .

We are left to show that $y_1, \dots, y_n$ are algebraically independent. As R is algebraic over these elements, there is a subset of them that form a transcendence basis

of R (Def 6.1.3 of the “Rings and modules” notes). However, all transcendence bases contain the same number of elements (Def 6.1.7 of the “Rings and modules” notes). Hence, $y_1, \dots, y_n$ is actually already algebraically independent. $\square$

Corollary 7.2.6. *In the situation of Theorem 7.2.5, we also have $p \cap k[y_2, \dots, y_n] = \{0\}$.*

Proof. Take $f \in p \cap k[y_2, \dots, y_n]$. Then $f \in Ry_1$, and hence the element in $\frac{f}{y_1} \in \text{Frac}(S) \subseteq \text{Frac}(R)$ is integral over S . However S is integrally closed (as it is a UFD, see Example 6.2.6 of the “Rings and modules” notes). Hence, $\text{Frac } f y_1 \in S$. This means that $f = y_1 g(y_1, \dots, y_n)$ for some polynomial g in n -variable over k . As $f \in k[y_2, \dots, y_n]$ by assumption, we obtain that $g = 0$, and hence $f = 0$. This concludes our proof. $\square$

Example 7.2.7. In the above proof we proved implicitly that if $S \subseteq R$ is an integral extension of domains with some $0 \neq a \in S$ and with S being integrally closed, then $Ra \cap S = Sa$. Indeed, take $f \in Ra \cap S$. Then $\frac{f}{a} \in \text{Frac } S$ is integral over S and hence $f \in Sa$.

This statement is false without assuming integral. Say if $S = k[x^2, x^3] \subseteq k[x] = R$ and $a = x^2$, then $Ra \cap S \ni x^3 \notin Sa$.

Lemma 7.2.8. *Consider the following situation:*

- R is a finitely generated k -algebra and also a domain, and
- $p \subseteq R$ is a prime ideal.
- $y_1, \dots, y_n \in R$ are algebraically independent elements such that $k[y_1, \dots, y_n] \cap p = \{0\}$.

Then $\dim(R/p) \geq n$

Proof. Let $\phi : R \rightarrow R/p$ be the quotient homomorphism. Let $z_i = \phi(y_i)$ for $i = 1, \dots, n$. We show that $z_1, \dots, z_n$ are algebraically independent elements of R/p . This will conclude our proof as we learned in “Rings and modules” that $\dim(R/p) = \text{trdeg}_k \text{Frac}(R/p)$ (Thm 6.1.11 in the “Rings and modules” notes). So, it is enough to exhibit $n - 1$ elements in R/p that are algebraically independent.

Suppose the opposite, that is, that $f \in k[x_1, \dots, x_n]$ gives an algebraic relation on z_i , that is $f(z_1, \dots, z_n) = 0$ in R/p . Then, we have $\phi(f(y_1, \dots, y_n)) = f(z_1, \dots, z_n) = 0$. Hence, $f(y_1, \dots, y_n) \in \ker \phi = p$. However, by our assumption the value of any polynomial evaluated in $y_1, \dots, y_n$ is in p if and only if this polynomial is 0. Hence, we have $f = 0$. $\square$

Proposition 7.2.9. *Consider the following situation:*

- R is a finitely generated k -algebra and also a domain, and
- $p \subseteq R$ is a prime ideal of height 1.

Then $\dim R = \dim(R/p) + 1$.

Proof. Set $n = \dim R$. By Lemma 7.2.4, we only have to show that $\dim(R/p) \geq n - 1$. By Lemma 7.2.8, it is enough to find algebraically independent elements $y_2, \dots, y_n \in R$ such that $k[y_2, \dots, y_n] \cap p = \{0\}$.

For this, let us start with finding an arbitrary Noether normalization $R' = k[z_1, \dots, z_n] \subseteq R$. If $k[z_1, \dots, z_n] \cap p = \{0\}$, then Lemma 7.2.8 shows that $\dim(R/p) \geq n$. However, then $\dim R \geq n + 1$, which is a contradiction. So, for $q = k[z_1, \dots, z_n] \cap p$ we have $q \neq \{0\}$. Corollary 7.2.6

then yields a subring $S = k[y_1, \dots, y_n] \subseteq R'$ generated by algebraically independent elements $y_1, \dots, y_n$, such that $k[y_2, \dots, y_n] \cap q = \{0\}$. However, this concludes the proof, as we have

$$k[y_2, \dots, y_n] \cap p \underset{\uparrow}{=} k[y_2, \dots, y_n] \cap R' \cap p = k[y_2, \dots, y_n] \cap q = \{0\}.$$

$k[y_2, \dots, y_n] \subseteq R'$

□

Proof of Theorem 7.2.1. Note that as R is a finitely generated k -algebra, all dimensions and heights are finite.

We show the statement by induction on $\text{ht } p$. If $\text{ht } p = 0$, then there is nothing to prove. If $\text{ht } p = 1$, then the statement is shown in Proposition 7.2.9.

For $h = \text{ht } p$, take a chain of prime ideals

$$p_0 = (0) \subsetneq \dots \subsetneq p_h = p \quad (7.2.c)$$

Such a chain exists by the definition of a prime ideal. By our induction hypothesis we know that

$$\dim R \underset{\uparrow}{=} \dim(R/p_{h-1}) + \text{ht } p_{h-1} \underset{\uparrow}{=} \dim(R/p_{h-1}) + h - 1 \underset{\uparrow}{=} \dim(R/p) + 1 + h - 1 = \dim(R/p) + \text{ht } p$$

induction hypothesis

$\text{ht } p_{h-1} \geq h - 1$ by (7.2.c) and it cannot be bigger, because then we would have $\text{ht } p_h > h$

Proposition 7.2.9 applied to R/p_{h-1} and to the image of p in R/p_{h-1}

Here we used that if q is the image of p in R/p_{h-1} , then $\text{ht } q = 1$ (as p_{h-1} is a maximal prime ideal contained in p_h), and by the third isomorphism theorem we have

$$R/p_{h-1} / q \cong R/p$$

□

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7.3 KRULL'S HAUPTIDEALSATZ

7.3.1 Minimal primes

Section 7.2 reduces the computation of the dimension of finitely generated algebras to heights of certain prime ideals. More precisely, if we know the dimension of a domain R that is also a finitely generated k -algebra, where k is a field, then

$$\dim(R/I) = \dim V(I) \underset{\substack{\uparrow \\ \text{as a subspace of } \text{Spec } R}}{=} \max_{\substack{p \supseteq I \text{ prime ideal} \\ \text{minimal with} \\ \text{this property}}} \dim V(p) \underset{\substack{\uparrow \\ \text{Theorem 7.2.1}}}{=} \max_{\substack{p \supseteq I \text{ prime ideal} \\ \text{minimal with} \\ \text{this property}}} \dim R - \text{ht } p \quad (7.3.a)$$

dimension of a Noetherian topological space is the maximum of the dimension of its irreducible components (easy exercise), which correspond to the minimal primes containing I

The irreducible components of $V(I)$ are the above $V(p)$, because the irreducible components are the maximal irreducible closed subsets of $V(I)$. This then on the ring level translates to being a prime ideal containing I . In fact, prime ideals of the above type are called minimal primes of I :

Definition 7.3.1. If r is an element (resp. I is an ideal) of a Noetherian ring R , then a prime ideal $p \subseteq R$ is a *minimal prime* of r (resp. of I), if $r \in p$ (resp. $I \subseteq p$), and p is minimal with respect to this property.

In the next lemma the notation $(I : x) := \{ r \in R \mid xr \in I \}$ is used where $x \in R$ and $I \subseteq R$ is an ideal. Recall also that an ideal $q \subseteq R$ is *primary* if $xy \in q$ and $q \not\subseteq q$ implies that there is an integer $n > 0$ such that $y^n \in q$ (Definition 8.5.1 from Rings and modules). Note that the radical of a primary ideal is prime (Proposition 8.5.3 from Rings and Modules).

Proposition 7.3.2. Let I be a radical ideal in a Noetherian ring R . Let $I = \bigcap_{i=1}^r q_i$ be a minimal primary decomposition of I . Then, q_i are primes and the set $\{q_i\}$ agrees with the set of minimal primes of I .

Proof. If $r = 1$, then I is prime. In particular, then there is nothing to prove, as I is its only minimal prime, and also its primary decomposition consists of I (by the unicity part of primary decomposition).

So, we may suppose that $r > 1$.

STEP 1: q_i are prime ideals. First we note:

Claim. For any $x \in R$, $(I : x)$ is a radical ideal.

Proof. Let $n > 0$ be an integer and $y \in R$. Then,

$$y^n \in (I : x) \Leftrightarrow y^n x \in I \Rightarrow y^n x^n \in I \xRightarrow{\substack{\uparrow \\ I \text{ is radical}}} xy \in I \Leftrightarrow y \in (I : x)$$

□

Note that for any $x \in R$, $q_i \subseteq (q_i : x)$ by definition. Fix an i . As the decomposition is minimal, we can find $x \in (\bigcap_{j \neq i} q_j) \setminus q_i$.

Claim. We claim that in fact $q_i = (q_i : x)$.

Proof. Suppose otherwise: there exists $r \in R$ such that $rx \in q_i$ but $r \notin q_i$. Then by definition of primary we have $x^n \in q_i$ for some n , and so $x^n \in I$ by the choice of x . But I is radical, so $x \in I$, and therefore $x \in q_i$, because $I \subset q_i$. This is a contradiction. □

Putting the above two claims one can obtain that q_i is prime as follows: for the other choice of x , we have

$$(I : x) = \left(\bigcap_j q_j : x \right) = \bigcap_j (q_j : x) \xRightarrow{\substack{\uparrow \\ (q_i : x) = q_i \text{ by the above claim, and } (q_j : x) = R \text{ for } j \neq i}} q_i$$

Hence, q_i is a primary ideal which is radical. In particular, q_i is prime.

STEP 2: q_i are minimal primes of I . By Thm 8.5.21 of the Rings and Modules notes, the set of primes which occur as radicals of the ideals in a minimal primary decomposition is unique. As we showed in Step 1 that q_i are primes, we obtain that q_i themselves are unique, and also by the virtue of coming from a minimal primary composition, none of them contains

the other. Suppose now there is some prime ideal p such that $I \subset p \subset q_i$. Then, we can replace q_i by p , contradicting the uniqueness of the q_i .

STEP 3: *All minimal primes of I appear among the q_i .* If $\{p_i\}$ is a finite set of distinct minimal primes of I then we cannot have $p_j \supseteq \bigcap_{i \neq j} p_i$ unless p_j contains one of the p_i . (Otherwise we could take elements $a_i \in p_i \setminus p_j$. Then $\prod_{i \neq j} a_i \in \bigcap_{i \neq j} p_i$ but cannot be in p_j by the prime property.) Now suppose p is a minimal prime of $I = \bigcap_i q_i$. As we have already showed that q_i is a minimal prime of I , the claim in the first sentence of Step 3 implies that p is one of the q_i . $\square$

Corollary 7.3.3. *If I is an ideal of a Noetherian ring R , then the minimal primes of I are exactly the ideals in a minimal primary decomposition of $\sqrt{I}$. In particular, there are finitely many of them.*

Proof. The primes containing I are exactly the same as the primes containing $\sqrt{I}$. Hence, the same holds for minimal primes. Then the statement was shown in Proposition 7.3.2. $\square$

7.3.2 Statement of Krull's hauptidealsatz and examples

Theorem 7.3.4 (Krull's Haptidealsatz). *If $0 \neq r \in R$ is an element of a Noetherian ring and p is minimal of r , then $\text{ht } p \leq 1$ [or equivalently if $V(p)$ is an irreducible component of $V(r)$, then $\text{codim } V(p) \leq 1$].*

Corollary 7.3.5. *If $0 \neq r \in R$ is an element of a Noetherian domain and p is minimal of r , then $\text{ht } p = 1$ [or equivalently if $V(p)$ is an irreducible component of $V(r)$, then $\text{codim } V(p) = 1$].*

Proof. This follows from Theorem 7.3.4 directly, taking into account that a non-zero prime ideal in a domain has height at least 1, as it contains as proper subset the prime ideal (0) . $\square$

Corollary 7.3.6. *If $X = \text{Spec } R$ for a finitely generated k -algebra R that is also a domain, and $0 \neq f \in R$, then $\dim V(f) = \dim \text{Spec } (R/(f)) = \dim X - 1$.*

Proof. This is just putting together (7.3.a) and Corollary 7.3.5. $\square$

Example 7.3.7. Let us assume that k is algebraically closed, and hence we can work both in the scheme theory and in the classical algebraic geometry world. Consider the subset

$$X = \overline{\{ (t^2, t^3, t^4) \in \mathbb{A}_k^3 \mid t \in k \}}.$$

We wrote the definition of X with the classical language, but using the functor t defined by Hartshorne in Prop 2.6, we identify the above definition with $t(X)$. That is, $t(X)$ contains the maximal ideals $(x - t^2, y - t^3, z - t^4)$ and all the other prime ideals in the closure of these.

We also note that certainly $\dim X \geq 1$, as it contains infinitely many points.

Let us find equations for X . Obviously $X \subseteq V(y^2 - xz)$. As in Example 7.2.3, we can show that $f = y^2 - xz \in k[x, y, z]$ is irreducible. So, $R = k[x, y, z]/(f)$ is a domain and $Y = \text{Spec } R$ is the variety defined by the equation f in $\mathbb{A}^3$. By the choice of f we also have that $X \subseteq Y$. However, as in Example 7.2.3, $\dim Y = 2$. As, X is defined by using a single parameter (the variable t), we expect it to be of dimension 1, and hence we expect that $Y \neq X$. With other words, there should be more equations describing X . For example, there is $g = x^2 - z$. This is also irreducible in $k[x, y, z]$, and as it is not associated to f , we have $g \notin (f)$. Hence the coset

$\bar{g}$ of g in R is a non-zero element. By [Corollary 7.3.5](#) and [Theorem 7.2.1](#), we obtain that every minimal prime of $\bar{g}$ in R has height 1 and hence

$$\dim \underbrace{V(f, g)}_{\substack{\uparrow \\ \text{in } \mathbb{A}^3}} = \dim \underbrace{V(\bar{g})}_{\substack{\uparrow \\ \text{in Spec } R \text{ considered as a topological subspace of } \mathbb{A}^3}} = 1.$$

So, we have a good chance that $V(f, g) = X$. In fact, if a closed point $(a, b, c) \in \mathbb{A}^3$ is in $V(f, g)$, then we have $c = a^2$ and then also $b = a^{3/2}$, where we have a choice of square-root. As we work over k algebraically closed this means that if we go with t through the two square roots of a , (t^2, t^3, t^4) will be going through all the possibilities of (a, b, c) . That is, $V(f, g) = X$ indeed.

To summarize, by using [Corollary 7.3.5](#) and [Theorem 7.2.1](#) as a helping heuristics to know how many equations we have to find, we showed that

$$X = V(x^2 - z, y^2 - xz).$$

7.3.3 Proof of Krull's hauptidealsatz

Material that we are skipping

The proof of [Theorem 7.3.4](#) uses some of the standard techniques of commutative algebra, in particular localization at a prime ideal. Recall that if p is a prime ideal of R , then there is an induced homomorphism $\iota : R \rightarrow R_p$, along which we extend and contract ideal (which are defined by $I^e = \iota(I) \cdot R_p$ and $J^c = \iota^{-1}(J)$). Also, [Proposition 6.19](#) or [Rings and modules](#) provides a precise description of these operations.

Lemma 7.3.8. *If R is a ring and $p \subseteq R$ is a prime ideal, then $\text{ht } p = \dim R_p$.*

Proof. This follows straight from the description of the prime ideals of R_p , that is, they correspond to prime ideals of R contained in p . $\square$

Lemma 7.3.9. *If $m^i = m^{i+1}$ in a Noetherian local ring (R, m) , then $\dim R = 0$.*

Proof. m^i is a finitely generated R -module such that $m \cdot m^i = m^i$. Therefore by Nakayama's lemma ([Sheet 9](#), exercise 1 of [Rings and Modules](#)), we have $m^i = 0$. In particular $m \subseteq \text{nil}(R)$, where $\text{nil}(R)$ is the nilradical. Now, let $q \subsetneq R$ be a prime ideal. Then, the following containments show that $q = m$:

$$\begin{array}{ccccc} m & \subseteq & \text{nil}(R) & \subseteq & q & \subseteq & m. \\ \uparrow & & \uparrow & & \uparrow & & \\ \boxed{\text{the above claim}} & & \boxed{q \text{ is prime}} & & \boxed{m \text{ is the only maximal ideal}} & & \end{array}$$

$\square$

Definition 7.3.10. If I is an ideal of a ring R , then the n -th symbolic power $I^{(n)}$ is

$$I^{(n)} = \{ r \in R \mid sr \in I^n \text{ for some } s \in R \setminus I \}.$$

Lemma 7.3.11. *If $p \subseteq R$ is a prime ideal in a ring, and $\iota : R \rightarrow R_p$ is the localization with maximal ideal $m \subseteq R_p$, then for each integer $n > 0$, $p^{(n)} = \iota^{-1}(m^n)$.*

Additionally, $p^{(n)} = p^{(n+1)}$ if and only if $m^n = m^{n+1}$.

Proof. As $m = pR_p$, $m^n = p^n R_p$, and hence $m^n = (p^n)^e$, and hence $\iota^{-1}(m^n) = (p^n)^{ec}$. By Proposition 7.3.9.(2) of the Rings and Modules notes, $(p^n)^{ec} = \bigcup_{u \in R \setminus p} (p^n : u)$,

which agrees with $p^{(n)}$ of Definition 7.3.10, by the definition of colon ideals.

In particular, the leftward implication of the addendum is automatic. For the other direction assume that $p^{(n)} = p^{(n+1)}$. By Proposition 7.3.9.(1) of the Rings and Modules notes, we obtain

$$\begin{array}{ccccccc}
 m^n & \stackrel{=}{=} & (m^n)^{ce} & \stackrel{=}{=} & (p^{(n)})^e & \stackrel{=}{=} & (p^{(n+1)})^e & \stackrel{=}{=} & (m^{n+1})^{ce} & \stackrel{=}{=} & m^{n+1} \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \boxed{\text{Proposition 7.3.9.(1) of}} & & \boxed{\text{the above}} & & \boxed{p^{(n)} = p^{(n+1)}} & & \boxed{\text{the above}} & & \boxed{\text{Proposition 7.3.9.(1) of}} \\
 \boxed{\text{the Rings and Modules}} & & \boxed{\text{paragraph}} & & & & \boxed{\text{paragraph}} & & \boxed{\text{the Rings and Modules}} \\
 \boxed{\text{notes}} & & & & & & & & \boxed{\text{notes}}
 \end{array}$$

□

Lemma 7.3.12. *If (R, m) is a Noetherian local ring such that $m^i = 0$ for some integer $i > 0$, then R is Artinian.*

Proof. Let K be the residue field R/m . The important thing to realize is that by Noetherianity m^j / m^{j+1} is a finite dimensional K -vector space for every j . In particular, these modules are Artinian R -modules. However, $0 = m^i \subseteq m^{i-1} \subseteq \cdots \subseteq m \subseteq R$ is a filtration of R by ideals such that the quotients of the adjacent terms in this filtration are Artinian R -modules. Then, by downward induction on j starting with $j = i$ one can prove that all m^j are Artinian, using the statement that if a submodule and the quotient by it are both Artinian then our module is Artinian (Prop 3.1.6 from the “Rings and modules” notes). □

Proof of Theorem 7.3.4. Let $q \subsetneq p$ be a prime ideal. We have to show that $\text{ht } q = 0$.

Claim. *We may assume that p is the only maximal ideal of R .*

Proof. What we actually show is that we may replace R , p , q and r by R_p , pR_p , qR_p and $\frac{r}{1} \in R_p$. Indeed:

- pR_p is the maximal ideal of the local ring R_p ,
- pR_p is a minimal prime ideal containing $\frac{r}{1}$, by the correspondence between prime ideals of the localization and prime ideals contained in p : Proposition 7.3.9.(4) of the “Rings and modules” notes, and
- by the same correspondence, we obtain that $qR_p \subsetneq pR_p$ is also a prime ideal of R_p , with $\text{ht } qR_p = \text{ht } q$.

□

Our goal showing that $\text{ht } q = 0$ is equivalent to showing that $\dim R_q = 0$, according to Lemma 7.3.8. Let $m := qR_q \subseteq R_q$ be the maximal ideal. According to Lemma 7.3.9, it is enough to show that $m^i = m^{i+1}$ for some integer i . For this, according to Lemma 7.3.11, it is enough to show that $q^{(i)} = q^{(i+1)}$. This is not easy

to show directly, as $q^{(1)} \supseteq q^{(2)} \supseteq \dots$ is a decreasing chain of ideals and there is no a-priori reason why a decreasing chain of ideals should stabilize in R . However, there is a ring floating around where we have such a situation. Before explaining this, first we need to clarify:

Claim. $\sqrt{(r)} = p$

Proof. According to [Proposition 7.3.2](#), $\text{rad}(r) = \bigcap_{i=1}^s p_i$ for the minimal primes p_i containing r , one of which agrees with p . Furthermore, no p_i can be thrown away from this decomposition. However, as p is the only maximal ideal, there cannot be any other prime ideal that is not contained in it. So, in fact $s = 1$, and $p_1 = p$. $\square$

As p is the unique maximal ideal in R , so is the image $\tilde{p}$ of p in $R/(r)$. Our latest claim implies that $\tilde{p}$ in $R/(r)$ is also nilpotent. Hence, according to [Lemma 7.3.12](#) every descending chain stabilizes in $R/(r)$. That is, we have found the desired ring. However, $q^{(i)}$ do not induce ideals of $R/(r)$. The best solution is to consider $(r) + q^{(i)}$, which do induce ideals of $R/(r)$ via the correspondence theorem. Hence this decreasing chain stabilizes for $i \geq s$. So, $q^{(s)} \subseteq (r) + q^{(s+1)}$. Therefore, any element $x \in q^{(s)}$ can be written as $x = yr + z$ for some $y \in R$ and $z \in q^{(s+1)} \subseteq q^{(s)}$. It follows that $yr \in q^{(s)}$ holds as well. However, $r \notin q$ (as p is the minimal prime ideal containing r).

Claim. We have $y \in q^{(s)}$.

Proof. We have the assumption $yr \in q^{(s)}$. That is, by [Definition 7.3.10](#), there is $\tilde{r} \in R \setminus q$ such that $yr\tilde{r} \in q^s$. However, as $r \notin q$ and as q is a prime ideal, $r\tilde{r} \notin q$ also holds. By using [Definition 7.3.10](#) and the containment $yr\tilde{r} \in q^s$, we obtain that $y \in q^{(s)}$ indeed. $\square$

So, by our latest claim, we obtained that every element of $q^{(s)}$ can be written as $yr + z$, where $y \in q^{(s)}$ and $z \in q^{(s+1)}$. In particular $q^{(s)} = q^{(s)}r + q^{(s+1)}$. So, $r \cdot M = M$, where M is the R -module $(q^{(s)}) / (q^{(s+1)})$. As $r \in p$, it also follows then that $p \cdot M = M$. Finally, by Nakayama lemma (Sheet 9, exercise 1 of Rings and Modules), $M = 0$, or equivalently $q^{(s)} = q^{(s+1)}$. This is exactly what we wanted to prove. $\square$

7.4 DIMENSION OF POLYNOMIAL RINGS OVER RINGS

Theorem 7.4.1. *If R is a Noetherian ring, then $\dim R[x_1, \dots, x_n] = n + \dim R$ [or equivalently $\dim \mathbb{A}_R^n = n + \dim R$].*

7.4.1 Height estimate in terms of the number of generators

Definition 7.4.2. For a general ideal I in a ring R , we define the height of I to be

$$\text{ht } I = \inf \{ \text{ht}(p) \mid I \subseteq p \subseteq R \text{ is prime ideal} \}$$

Note that here we may restrict to minimal primes of I . If R is Noetherian, then there are finitely many of those, and hence this infimum is a minimum.

Theorem 7.4.3. *If R is a Noetherian ring, and $r_1, \dots, r_s \in R$, then $\text{ht}(r_1, \dots, r_s) \leq s$ [or equivalently $\text{codim } V(r_1, \dots, r_s) \leq s$ in $\text{Spec } R$].*

The hard part about proving **Theorem 7.4.3** is to deal with the case when the chain giving the height of a minimal prime of $(r_1, \dots, r_s)$ does not contain the minimal primes of $(r_1, \dots, r_{s-1})$. **Lemma 7.4.5** says that in fact, one can always change the r_i such that this does not happen, at least in the case of a local ring (to which we will be able to reduce our argument by localization). Before **Lemma 7.4.5** we also state a direct consequence of Krull's hauptidealsatz (**Theorem 7.3.4**).

We call a prime ideal q contained in another prime ideal p *maximal in p* if $q \neq p$, and if $q \subseteq I \subseteq p$ is another prime ideal, then $I = q$ or $I = p$.

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Lemma 7.4.4. *If $q \subsetneq p \subseteq R$ are prime ideals of R , and $r \in p \setminus q$ such that p is a minimal prime containing both q and r , then q is maximal in p .*

Proof. Apply **Theorem 7.3.4** to $r + q \in R/q$. □

Lemma 7.4.5. *Let (R, m) be a local ring, and $r_1, \dots, r_s \in m$ such that $\sqrt{(r_1, \dots, r_s)} = m$ [or equivalently $V(r_1, \dots, r_s) = m \in \text{Spec } R$]. If $p \subsetneq m$ is a maximal prime in m , then there are elements $r'_1, \dots, r'_s \in m$ such that*

- (1) $\sqrt{(r'_1, \dots, r'_s)} = m$ [or equivalently $V(r'_1, \dots, r'_s) = m \in \text{Spec } R$], and
- (2) p is a minimal prime of $(r'_1, \dots, r'_{s-1})$ [or equivalently $V(p)$ is an irreducible component of $V(r'_1, \dots, r'_s)$].

Proof. As m is the only prime ideal containing all the r_i , there is an $r_i \notin p$. By re-indexing we may assume that $r_s \notin p$. Then we have

$$p \subsetneq p + (r_s) \subseteq \sqrt{p + (r_s)} \stackrel{\uparrow}{=} m,$$

the primary decomposition of $\sqrt{p + (r_s)}$ consists of only prime ideals, and it can contain only p as $p \neq p + (r_s)$.

Hence, we may find an integer $n > 0$ such that $r_i^n \in p + (r_s)$ for every integer $1 \leq i \leq s-1$. With other words, for every integer $1 \leq i \leq s-1$, we may define $r'_i \in p$ and $a_i \in R$ such that

$$\forall 1 \leq i \leq s-1: r_i^n = r'_i + a_i r_s \tag{7.4.a}$$

We also set $r'_s = r_s$. Then, we prove the two points of the lemma, one by one:

(1) we have

$$\begin{array}{ccccc} m & \stackrel{\uparrow}{=} & \sqrt{r_1, \dots, r_s} & \subseteq & \sqrt{r'_1, \dots, r'_s} & \subseteq & m \\ & \uparrow & & \uparrow & & \uparrow & \\ \boxed{\text{assumption}} & & \boxed{r_i \in \sqrt{(r'_1, \dots, r'_s)} \text{ by (7.4.a)}} & & \boxed{r'_i \in m \text{ by definition}} \end{array}$$

- (2) By the definition of the r'_i we have $p \supseteq (r'_1, \dots, r'_{s-1})$. So, it is only left to show that p is minimal with this property. So, assume the opposite, that is, that there is another prime ideal $(r'_1, \dots, r'_{s-1}) \subseteq q \subseteq p$. As m is a minimal prime containing $(r'_1, \dots, r'_{s-1}, r'_s)$, it is also a minimal prime containing q and r'_s . Hence by Lemma 7.4.4, q is maximal in m . However, this contradicts the fact that $q \subsetneq p \subsetneq m$.

□

Proof of Theorem 7.4.3. We prove the statement by induction on s . For $s = 1$ the statement is shown by Krull's hauptidealsatz (Theorem 7.3.4). So, we assume that $s > 1$, and that we know the statement of the theorem for smaller values of s .

Consider a minimal prime p of $(r_1, \dots, r_s)$. It is enough to prove that $\text{ht } p \leq s$. For that we can localize at p , or with other words we may assume that p is the unique maximal ideal of R . (One needs to check here that the assumptions of the theorem do not change, and also the height of p does not change. We let you do this checking, which is very similar to the checkings we did in the proof of Theorem 7.3.4.)

Let $q \subseteq p$ be a prime ideal such that q is maximal in p . By applying Lemma 7.4.5, we may assume that q is a minimal prime of $(r_1, \dots, r_{s-1})$. By our induction hypothesis $\text{ht } q \leq s - 1$. As this is true for all prime ideals that are maximal in p , we obtain that $\text{ht } p \leq s$.

□

Corollary 7.4.6. *If $I \subseteq R$ is an ideal of a Noetherian ring, then $\text{ht } I < \infty$.*

Proof. As R is Noetherian, we have $I = (r_1, \dots, r_s)$, and hence $\text{ht } I \leq s$ [or equivalently $\text{codim } I < \infty$ in $\text{Spec } R$].

□

Corollary 7.4.7. *(R, m) is a Noetherian local ring, then $\dim R < \infty$ [or equivalently $\dim \text{Spec } R < \infty$].*

Proof. By definition, $\dim R = \text{ht } m$, and then we apply Corollary 7.4.6.

□

7.4.2 Height of general complete intersections

Theorem 7.4.8. *Consider the following situation:*

- (1) A is a Noetherian ring,
- (2) $0 \leq j \leq i$ are integers,
- (3) $I \subseteq A$ is an ideal with $\text{ht}(I) \geq i$ [or equivalently $\text{codim } V(I) \geq i$ in $\text{Spec } R$]
- (4) $r_1, \dots, r_j \in I$ such that $\text{ht}(r_1, \dots, r_j) = j$ [or equivalently $V(r_1, \dots, r_j)$ has codimension j in $\text{Spec } R$, note that $V(r_1, \dots, r_j) \supseteq V(I)$]

Then, there is $r_{j+1}, \dots, r_i \in I$ so that $\text{ht}(r_1, \dots, r_s) = s$ for every $j \leq s \leq i$ [or equivalently $\text{codim}(r_1, \dots, r_s) = s$].

$V(r_1, \dots, r_s)$ as in the statement of Theorem 7.4.8, that is, satisfying the condition $\text{ht}(r_1, \dots, r_s) = s$ are called set theoretic complete intersections. The geometric meaning of Theorem 7.4.8 is that if an ideal I defines a codimension at least i closed subset, then every set theoretic complete intersection formed of elements of I can be extended to a codimension i set theoretic complete intersections using elements of I .

The proof itself is just a combination of the corollary of HAUPTIDEALSATZ stated in Lemma 7.4.4, and of the following lemma.

Lemma 7.4.9. PRIME AVOIDANCE. *If $p_1, \dots, p_s$ are prime ideal in a ring R , and $I \subseteq R$ is an ideal such that $I \not\subseteq p_i$ for every $1 \leq i \leq s$ [or equivalently I does not vanish completely on $V(p_i)$]. Then $I \not\subseteq \bigcup_i p_i$ [or equivalently there is an element of I that does not vanish on any of the $V(p_i)$].*

Proof. We may assume that no two the p_i 's are contained in each others. That is, for each $i \neq j$ we have an element $r_{ji} \in p_j \setminus p_i$. Then by the prime property we have $b_j = \prod_{i \neq j} r_{ji} \in \left(\bigcup_{j \neq i} p_j \right) \setminus p_i$.

By assumption $I \not\subseteq p_i$ for each i we also have $a_i \in I \setminus p_i$. Again by prime property $b_i a_i \in I \cap \left(\bigcup_{j \neq i} p_j \right) \setminus p_j$. Then

$$\sum_{i=1}^s b_i a_i \in I \setminus \left(\bigcup_{i=1}^s p_i \right).$$

□

Proof of Theorem 7.4.8. The statement is made up so that, by induction, it is enough to prove the case $j = i - 1$. Let $p_1, \dots, p_l$ be the minimal primes of $(r_1, \dots, r_j)$. By Theorem 7.4.3, and by our assumption $\text{ht } p_n = j$ for all n . Hence, $I \not\subseteq p_n$ for every n . By Lemma 7.4.4, we can choose $r_{j+1} \in I \setminus \left(\bigcup_{i=1}^l p_i \right)$. Let q be a mimal prime of $(r_1, \dots, r_{j+1})$. As $(r_1, \dots, r_j) \subseteq q$, q contains one of the minimal primes of $(r_1, \dots, r_j)$, that is one of the p_n 's. Additionally by our choice of r_{j+1} this containment is not trivial. Hence $\text{ht}(r_1, \dots, r_{j+1}) \geq j + 1 = i$. We already know the reverse inequality as well, by Theorem 7.4.3. This concludes our proof. □

7.4.3 Dimension of local rings

Corollary 7.4.10. *If (R, m) is a Noetherian local ring, then*

$$\dim R = \min \left\{ n \in \mathbb{Z}^{>0} \mid \sqrt{(r_1, \dots, r_n)} = m \right\}$$

Proof. By definition $\dim R = \text{ht } m$. Then, we just combine Theorem 7.4.3 and Theorem 7.4.8, noting that for an ideal $I \subseteq m$, $\text{ht } I = \text{ht } m$ if and only if m is the only minimal prime of I . □

Definition 7.4.11. If (R, m) is a Noetherian local ring, then a *system of parameters* is a collection of elements $r_1, \dots, r_{\dim R} \in m$ such that $\sqrt{(r_1, \dots, r_{\dim R})} = m$ [or equivalently $V(r_1, \dots, r_{\dim R}) = m \in \text{Spec } R$].

7.4.4 Proof of Theorem 7.4.1

Proof of Theorem 7.4.1. We may assume that $s = \dim R < \infty$, and that $n = 1$. We also use the notation x instead of x_1 .

dim $R[x] \geq s + 1$: Take then a chain $p_0 \subsetneq \dots \subsetneq p_s$ exhibiting this. Then,

$$R[x] \cdot p_0 \subsetneq \dots \subsetneq R[x]p_s \subsetneq R[x]p_s + R[x] \cdot x$$

is a chain showing that $\dim R[x] \geq n + 1$. We note that $R[x] \cdot p_i \cong R[x] \otimes_R p_i$ by the flatness of $R[x]$ over R . This shows that $R[x] \cdot p_{i+1} / R[x] \cdot p_i \cong R[x] \otimes_R (p_{i+1}/p_i)$, and hence all the steps in the above chain, except the last one are not equalities. For the last one, being not an equality one just realizes that $R[x] / R[x] \cdot p_s \cong R[x] \otimes_R (R/p_s)$, and hence the class of x is not zero in this quotient ring.

dim $R[x] \leq s + 1$: For this we may assume that R is local with maximal ideal m , and $\dim R \leq s$. Then all maximal ideals of $R[x]$ are of the form $m' = R[x] \cdot m + R[x] \cdot f$, such that the coset

of f in $(R/m)[x]$ is an irreducible polynomial. However, if $r_1, \dots, r_j$ is a system of parameters for R , then $r_1, \dots, r_{\dim R}, f$ yield elements in $R[x]_{m'}$ such that $V(r_1, \dots, r_{\dim R}, f) = m'$. Then we just apply [Corollary 7.4.10](#) to get that $\dim R[x]_{m'} \leq \dim R + 1 \leq s + 1$, and then also $\dim R[x] \leq s + 1$.

□

Chapter 8

Projective schemes

8.1 GENERAL REMARKS AND ADDITIONS TO THE MATERIAL

Here we study pages 76-77 of Hartshorne that associates to a graded ring S the scheme $\text{Proj } S$.

Remark 8.1.1. It is extremely important to note exercise II.2.14.(d) from Hartshorne which says that if there is a classical projective variety V in $\mathbb{P}^n$ given by the homogeneous ideal $I(V)$, and hence having homogeneous coordinate ring $S(V) = k[x_0, \dots, x_n]/I(V)$, the scheme $\text{Proj } S(V)$ is the scheme corresponding to V , or formally $\text{Proj } S(V) = t(V)$.

For example, the (classical) elliptic curve in $\mathbb{P}^2$ given by $zy^2 = x^3 - xz^2$ is then identified with the scheme $\text{Proj } (k[x_0, x_1, x_0]/(zy^2 = x^3 - xz^2))$.

Remark 8.1.2. The use of *graded ring* in Hartshorne is a bit ambiguous. A priori by a graded ring he means a $\mathbb{N}$ -graded ring, that is $S = \bigoplus_{i \in \mathbb{N}} S_i$. However, as soon as one localizes these rings, one needs to allow $\mathbb{Z}$ -graded rings, which is essential in the proofs.

So, to sum it up, the statements are for $\mathbb{N}$ -graded rings, but the proofs and constructions do use $\mathbb{Z}$ -graded rings. So, it is best to always say what grading one uses. Also note that a $\mathbb{N}$ -graded ring is also $\mathbb{Z}$ -graded.

To be really precise: the definition of $\text{Proj } S$, and Lemma 2.4 and Proposition 2.5 of Hartshorne are for $\mathbb{N}$ -graded rings (for example because the definition of $\text{Proj } S$ uses the *irrelevant ideal* S_+ which is not an ideal in the $\mathbb{Z}$ -graded case). However, the proofs use $\mathbb{Z}$ -graded rings.

Remark 8.1.3. Hartshorne defines $S_{(p)}$ (here $p \subseteq S$ is a homogeneous ideal) as the degree zero part in the localization of S by

$$T = (S \setminus p)_{\text{hom}} = \{ f \in S \setminus p \mid f \text{ is a homogeneous element} \}$$

This is a $\mathbb{Z}$ -graded ring, before passing to the degree 0-part. It is important to note here that by the definition of graded rings the degree 0 part is always a subring.

Similarly, Hartshorne defines $S_{(f)}$ to be (here $f \in S$ is homogeneous) the degree 0 part of S_f . This is a *very unfortunate* notation, as it also denotes the localization at the prime ideal (f) . Be careful with it!

Note that in a $\mathbb{Z}$ -graded ring S the degree d -part S_d is always a module over the degree zero part S_0 . Then the following proposition does make sense:

Proposition 8.1.4. *Let S be a $\mathbb{Z}$ -graded ring such that there is an invertible element $f \in S_d$.*

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Then for every integer $s > 0$ we have the following isomorphism of S_0 -modules:

$$\begin{array}{ccc} S_0 & \xleftarrow{\cong} & S_{sd} \\ \Psi & & \Psi \\ r & \longmapsto & f^d r \\ & & \\ r' f^{-d} & \longleftarrow & r' \end{array}$$

With other words we have a ring isomorphism

$$\underbrace{S^{(d)}}_{\substack{\uparrow \\ d\text{-the Veronese subring}}} \stackrel{\text{def}}{=} \bigoplus_{i \in \mathbb{Z}} S_{id} \cong S_0[x, x^{-1}]$$

Proof. HW □

Example 8.1.5. Let's consider $k[x, y]$ with the standard grading (so $\deg x = \deg y = 1$). Then,

$$k[x, y]_{(y)} \cong (k[x][y, y^{-1}])_0 = k\left[\frac{x}{y}, y, y^{-1}\right]_0 = k\left[\frac{x}{y}\right]$$

Similarly

$$k[x_0, \dots, x_n]_{(x_i)} \cong (k[x_0, \dots, x_n, x_i^{-1}])_0 \cong k\left[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right].$$

IMPORTANT: the above isomorphism is given by homogenization and dehomogenization, so we have seen it in disguise in classical algebraic geometry.

The next definition matches up the properties of the Proj construction learned in Proposition 2.5 of Hartshorne and classical algebraic geometry. It tells you that the description $\text{Proj } S|_{D_+(x_i)} \cong \text{Spec } S_{(x_i)}$ is compatible with the way we were constructing affine charts to classical projective varieties, via the functor t of Proposition 2.6 of Hartshorne.

Proposition 8.1.6. Let $S = k[x_0, \dots, x_n]/I$, where

- $k[x_0, \dots, x_n]$ has the standard grading,
- $I \subseteq k[x_0, \dots, x_n]$ is a homogeneous ideal such that $I_0 = I_1 = 0$ [if k is algebraically closed this means that it defines a projective algebraic set that is not contained in any hyperplanes]

Let $\phi : k[x_0, \dots, x_n] \rightarrow k[x_0, \dots, x_n]_{x_i} \cong k[x_0, \dots, x_n, x_i^{-1}]$ the localization map, and let extension and contraction of ideals denote extension and contraction via this map. Then:

- (1) The ideal $(I^e)_0 \subseteq k[x_0, \dots, x_n]_{(x_i)} \cong k\left[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right]$ is generated by the dehomogenization of any generator set of I .
- (2) Homogenization and dehomogenization gives isomorphism

$$S_{(x_i)} \cong k[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]/(I^e)_0$$

So, when S is finitely generated over a field k by homogeneous elements of degree 1, then one can deduce from the above proposition that (at least if k is algebraically closed), then we are in the classical situation, we have $\text{Proj } S = t(V)$ for an appropriate projective variety V (exercise II.2.14.(d) of Hartshorne is about putting together the details of this).

A non-classical situation is the following:

Example 8.1.7. Let's take $S = k[x, y, z]$ with a non-standard grading, for which $\deg x = 2$ and $\deg y = \deg z = 1$. Then

$$S_x = k \left[\underbrace{\frac{y^2}{x}, \frac{z^2}{x}, \frac{yz}{x}}_{\substack{\uparrow \\ \text{generated subring, not the polynomial ring over} \\ \text{these elements}}} \right] \cong k[u, v, w] / (uv - w^2)$$

$$\boxed{u \leftrightarrow \frac{y^2}{x}, v \leftrightarrow \frac{z^2}{x}, w \leftrightarrow \frac{yz}{x}}$$

With other words $D_+(x) \cong \text{Spec} \left(k[u, v, w] / (uv - w^2) \right) \not\cong \mathbb{A}^2$. To see this final non-isomorphism one just shows that the local ring of $k[u, v, w] / (uv - w^2)$ at (u, v, w) is not regular, which notion is defined below.

Definition 8.1.8. A Noetherian local ring (R, m) is regular if $\dim R = \dim_k (m/m^2)$, where $k := R/m$.

A point x of a scheme X is regular if $\mathcal{O}_{X,x}$ is regular. The variety X itself is regular, if all its points are regular. The opposite of regular is called *singular* both for points and for varieties.

8.2 PROOF OF PROPOSITION 2.5 OF HARTSHORNE

To understand Proposition 2.5 of Hartshorne it is really important to understand that we can at the beginning pass to an Veronese subring. For that we need to understand that that way Proj does not change, by this:

Proposition 8.2.1. *If S is an $\mathbb{N}$ -graded ring, and $d > 0$ an integer, then $\text{Proj } S^{(d)} \cong \text{Proj } S$ naturally.*

Proof. We leave the details to the reader, we mention only a few parts of the construction of the isomorphism. Let $\phi : S^{(d)} \hookrightarrow S$ be the inclusion. Then, the following operations give a bijection between the radical ideals of $S^{(d)}$ and S :

- (1) $S^{(d)} \supseteq I \mapsto \sqrt{I^e} \subseteq S$
- (2) $S \supseteq J \mapsto \phi^{-1}J \subseteq S^{(d)}.$

In particular, as radical ideals define the topology we obtain that the underlying topological spaces of $\text{Proj } S^{(d)}$ and of $\text{Proj } S$ are isomorphic.

Let $p \subseteq S^{(d)}$ and $q \subseteq S$ be prime ideals corresponding to each others via this correspondence. Then, there is a natural induced ring homomorphism $S_p^{(d)} \rightarrow S_q$. Then, one shows that the induced map $S_{(p)}^{(d)} \rightarrow S_{(q)}$ is an isomorphism. $\square$

It is very much suggested to supplement/amend the beginning of the proof of Proposition 2.5.(b) of Hartshorne by the following: let $d = \deg f$. By **Proposition 8.2.1** we may replace S by $S^{(d)}$, and then we can replace the grading, which is now by $d\mathbb{N}$, by dividing each degree by d so that $\deg f = 1$. With other words, with this addition to the proof, one can assume that $\deg f = 1$.

If one does not make this addition to the proof, then one gets into cumbersome analysis of relating S_f and $(S_f)_0$. Indeed, as now we assumed that $\deg f = 1$, by **Proposition 8.1.4**, we have a homomorphism as follows when $p \in D_+(f)$:

$$S_f \cong S_{(f)}[x, x^{-1}] \xrightarrow[\substack{\text{graded homomorphism, that is,} \\ \text{it respects the grading, i.e., it} \\ \text{sends } x \text{ to } x}]{\text{}} S_p \cong S_{(p)}[x, x^{-1}]$$

In particular, we see that homogeneous ideals of S_f are all of the form $I[x, x^{-1}]$ where I is an ideal of $S_{(f)}$. Hence, the topology of $D_+(f)$ and of $\text{Spec } S_{(f)}$ is “the same”. More precisely, if $\phi : S \rightarrow S_f$ is the localization homomorphism, then the bijection is as follows:

$$\begin{array}{ccc} D_+(f) & \longleftrightarrow & \text{Spec } S_f \\ \Psi & & \Psi \\ I & \longmapsto & (\phi(I))_0 \end{array}$$

$$\phi^{-1}(J[x, x^{-1}]) \longleftrightarrow J$$

Consider now for a fixed $p \in D_+(f)$ the homomorphism $S_{(f)} \rightarrow S_{(p)}$. Note that $(\phi(p))_0$ is exactly the preimage of the maximal ideal of $S_{(p)}$. Hence, by the universal property of localization, we obtain a commutative diagram as follows

$$\begin{array}{ccccc} & & \text{---} & & \\ & \searrow & & \nearrow & \\ S_{(f)} & \longrightarrow & (S_{(f)})_{(\phi(p))_0} & \xrightarrow{\alpha} & S_{(p)} \end{array}$$

Then one has to prove that α is an isomorphism. For this it is useful to see that

$$S_{(f)} \setminus (\phi(p))_0 = \left\{ \frac{a}{f^n} \in S_{(f)} \mid a \in S_n, \nexists m > 0 : af^m \in p \right\} \underset{\substack{\uparrow \\ f \notin p, p \text{ is prime}}}{=} \left\{ \frac{a}{f^n} \in S_{(f)} \mid a \in S_n, a \notin p \right\}$$

That is,

$$(S_{(f)})_{(\phi(p))_0} = \left\{ \frac{\frac{a}{f^n}}{\frac{b}{f^m}} \mid a \in S_n, b \in (S \setminus p)_m \right\}$$

and

$$S_{(p)} = \left\{ \frac{a}{b} \mid \exists n > 0 : a, b \in S_n, b \notin p \right\}$$

where both sets are quotiented by the appropriate equivalence relation. Additionally we have

$$\alpha \left(\frac{\frac{a}{f^n}}{\frac{b}{f^m}} \right) = \frac{af^m}{bf^n}$$

We explain the other details of the proof (including that α is an isomorphism) during the lectures, or we leave them homework.

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8.3 REGULARITY OF CLASSICAL VARIETIES

Lemma 8.3.1. *If $\lambda : S \twoheadrightarrow R$ is a surjective ring homomorphism, and $I \subseteq S$ is an ideal, then $I^{ec} = I + \ker \lambda$.*

Remark 8.3.2. Note that by the surjectivity assumption $I^e = \lambda(I)$, as if $a \in I$, and $r \in R$, then there exists $s \in S$, such that $\lambda(s) = r$, and we have $r\lambda(a) = \lambda(s)\lambda(a) = \lambda(sa)$, where $sa \in I$.

Proof. Obviously, $I^{ec} \supseteq I + \ker \lambda$, so take $a \in I^{ec}$. Then there exists $b \in I$, such that $\lambda(b) = \lambda(a)$, and hence $a - b \in \ker \lambda$. $\square$

Lemma 8.3.3. *Consider the following situation:*

- $P = (c_1, \dots, c_n) \in k^n$,
- $\tilde{m} = (x_1 - c_1, \dots, x_n - c_n) \subseteq k[x_1, \dots, x_n]$ the maximal ideal corresponding to P ,
- $I \subseteq k[x_1, \dots, x_n]$ is an ideal such that $I \subseteq \tilde{m}$ [or equivalently, $P \in V(I)$]
- $R = k[x_1, \dots, x_n] / I$, and
- $m \subseteq R$ is the image of $\tilde{m}$.

Then

(1) The map

$$\phi : k[x_1, \dots, x_n] \ni f \mapsto \begin{bmatrix} \frac{\partial f}{\partial x_1}(P) \\ \vdots \\ \frac{\partial f}{\partial x_n}(P) \end{bmatrix} \in k^n.$$

induces a k -linear isomorphism $\xi : \tilde{m} / \tilde{m}^2 \rightarrow k^n$.

(2) We have k -linear isomorphisms

$$k^n / \phi(I) \xleftarrow[\cong]{} \tilde{m} / (\tilde{m}^2 + I) \xrightarrow[\cong]{} m / m^2,$$

where the former is induced by ξ and the latter is induced by the natural map $\tilde{m} \rightarrow m$.

Proof. We note that the residue classes of $x_i - c_i$ form a basis of $\tilde{m} / \tilde{m}^2$ (one can prove the easiest by translating P to the origin).

(1) First, we note that ϕ is k -linear by the definition of the algebraic partial derivatives.

Second, we claim that $\phi(\tilde{m}^2) = 0$. Indeed, a general element of $\tilde{m}^2$ is of the form $f = \sum_{i \leq j} r_{ij}(x_i - c_i)(x_j - c_j)$ for some $r_{ij} \in k[x_1, \dots, x_n]$. Furthermore, for any such element $\phi(f) = 0$, as for every l, i, j , the partial derivative $\frac{\partial}{\partial x_l} r_{ij}(x_i - c_i)(x_j - c_j)$ consists of three products and each product contains $x_i - c_i$ or $x_j - c_j$ as a factor, which takes value 0 at P .

Hence, the above claim shows that ϕ induces a k -linear homomorphism $\xi : \tilde{m} / \tilde{m}^2 \rightarrow k^n$. As

$$\left\{ \phi(x_i - c_i) \mid i = 1, \dots, n \right\} = \left\{ [0, \dots, 0, \underset{\substack{\uparrow \\ \boxed{i-1\text{-th}}}}{1}, \underset{\substack{\uparrow \\ \boxed{i\text{-th}}}}{0}, \dots, 0]^T \mid i = 1, \dots, n \right\}$$

form a basis of k^n , ξ takes a basis to a basis, and hence it is an isomorphism.

(2) Let $\alpha : \tilde{m} / \tilde{m}^2 \rightarrow m / m^2$ be the natural homomorphism induced by the surjection $\tilde{m} \twoheadrightarrow m$. Using [Lemma 8.3.1](#) and that $I \subseteq \tilde{m}$, we obtain that ξ induces an isomorphism

$$\tilde{m} / \tilde{m}^2 \supseteq (I + \tilde{m}^2) / \tilde{m}^2 \rightarrow \phi(I).$$

However, by again using [Lemma 8.3.1](#), $(I + \tilde{m}^2) / \tilde{m}^2 = \ker \alpha$. Hence, the isomorphisms of the statements are obtained by quotienting out by $\ker \alpha$ in $\tilde{m} / \tilde{m}^2$ and by the image of

$\ker \alpha$ in k^n via ξ . This is pictured in the following commutative diagram, the columns of which are exact:

$$\begin{array}{ccccc}
 \phi(I) & \xleftarrow{\cong} & I + \tilde{m}^2 / \tilde{m}^2 = \ker \alpha & & \\
 \downarrow & & \downarrow & & \\
 k^n & \xleftarrow{\cong} & \tilde{m} / \tilde{m}^2 & \xrightarrow{\alpha} & m / m^2 \\
 \downarrow & & \downarrow & & \parallel \\
 k^n / \phi(I) & \xleftarrow{\cong} & \tilde{m} / (\tilde{m}^2 + I) & \xrightarrow{\cong} & m / m^2
 \end{array}$$

□

Proposition 8.3.4. *Consider the following situation*

- $X = \operatorname{Spec} R$ for $R = k[x_1, \dots, x_n] / I$, where $I = (f_1, \dots, f_r)$,
- $P = (c_1, \dots, c_n) \in k^n$, and m is the image in R of the ideal $\tilde{m}$ where $I \subseteq \tilde{m} = (x_1 - c_1, \dots, x_n - c_n) \subseteq k[x_1, \dots, x_n]$.

Then $m \in X$ is regular if and only if the rank of the matrix $\left[\frac{\partial f_i}{\partial x_j}(P) \right]_{j=1, \dots, n; i=1, \dots, r}$ is $n - \dim X$.

Proof. We use the notations of [Lemma 8.3.3](#) throughout the proof. Let A be the matrix $\left[\frac{\partial f_i}{\partial x_j}(P) \right]_{j=1, \dots, n; i=1, \dots, r}$. For a general element $f := \sum_{i=1}^r s_i f_i \in I$,

$$\frac{\partial f}{\partial x_j}(P) = \frac{\partial (\sum_{i=1}^r s_i f_i)}{\partial x_j}(P) \underset{\substack{\text{additivity of derivation and} \\ \text{Leibnitz rule}}}{=} \sum_{i=1}^r \frac{\partial s_i}{\partial x_j}(P) f_i(P) + s_i(P) \frac{\partial f_i}{\partial x_j}(P) \underset{\substack{f_i \in I, I \subseteq \tilde{m} \Rightarrow f_i(P) = 0}}{=} \sum_{i=1}^r s_i(P) \frac{\partial f_i}{\partial x_j}(P). \quad (8.3.a)$$

Hence:

$$\phi(f) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(P) \\ \vdots \\ \frac{\partial f}{\partial x_n}(P) \end{bmatrix} \underset{(8.3.a)}{=} \sum_{i=1}^r s_i(P) \begin{bmatrix} \frac{\partial f_i}{\partial x_1}(P) \\ \vdots \\ \frac{\partial f_i}{\partial x_n}(P) \end{bmatrix} \underset{\substack{\text{i-th column of } A}}{\uparrow}. \quad (8.3.b)$$

As s_i can be chosen so that $(s_i(P) \mid i = 1, \dots, r) \in k^r$ takes any value, (8.3.b) tells us that $\phi(I) = \operatorname{im} A$. Hence, $\operatorname{rk} A = \dim \phi(I)$. Now, consider the equalities

$$\dim_k(m/m^2) \underset{\substack{\uparrow \\ \text{Lemma 8.3.3.(2)}}}{=} n - \dim \phi(I) = n - \operatorname{rk} A.$$

So, $\dim_k(m/m^2) = \dim X$ if and only if the rank of A is $n - \dim X$. Additionally by [Example 6.0.9](#) and [Remark 6.0.10](#), we have that $m/m^2 \cong mR_m/m^2R_m = \bar{m}/\bar{m}^2$, where $\bar{m}$ is the maximal ideal of $\mathcal{O}_{X,m}$. This concludes our proof. □

Example 8.3.5. Let k be algebraically closed. We show that $X = \operatorname{Spec} (k[x, y] / (y^2 - x^3))$ is regular at all closed point except at $(0, 0)$. By [Proposition 8.3.4](#) we have to show that the following matrix has rank $2 - 1 = 1$ for all closed points $P \in X \setminus \{(0, 0)\}$ (as usual when we

work with schemes that can be identified with classical varieties, we identify the maximal ideals and the corresponding traditional points):

$$A := \begin{bmatrix} \frac{\partial(y^2-x^3)}{\partial x} \\ \frac{\partial(y^2-x^3)}{\partial y} \end{bmatrix} = \begin{bmatrix} -3x^2 \\ 2y \end{bmatrix}$$

Indeed, as A is a 2×1 -matrix, $A(P)$ having rank 1 means that it is not zero. So, fix $X \ni P = (x, y) \neq (0, 0)$. By the equation $y^2 - x^3 = 0$ we obtain that in fact both x and y are non-zero. We are not quite ready here, because the entries of $A(P)$ could still be zero, if $\text{char } k = 2$ or $\text{char } k = 3$. But in either case, at least one of them is non-zero, so $A(P)$ itself is non-zero.

One shows similarly that $(0, 0, 0)$ is the only singular closed point of $\text{Spec } \left(k[u, v, w] / (u^2 - vw) \right)$.

Chapter 9

Properties of schemes

This is about Section II.3 of Hartshorne. There are quite a few definitions at the beginning of the section that are important to know, including examples (reduced/non-reduced, integral, Noetherian schemes).

An important non-Noetherian example is:

Example 9.0.1. $X = \operatorname{Spec} R$, where $R = \mathbb{F}_p[x^{1/p^\infty}] = \bigcup_{n \geq 0} \mathbb{F}_p[x^{1/p^n}]$ where the union is taken in a fixed algebraic closure of $\mathbb{F}_p(x)$.

The first important statement is Prop 3.2 of Hartshorne. In the proof the following lemma is used:

Lemma 9.0.2. *If R is a Noetherian ring and $S \subseteq R$ is a multiplicatively closed set. Then $S^{-1}R$ is also Noetherian.*

Proof. Let us consider extension and contraction via the localization map. Let $I \subseteq S^{-1}R$ be an ideal. Then $I^c = (f_1, \dots, f_r)_R$ for some $f_i \in R$ as R is Noetherian. However, then $I = I^{ce} = (f_1, \dots, f_r)_{S^{-1}R}$. $\square$

It is particularly important to remember the reduction steps of Prop 3.2 to be able to solve next week's homework problems. The main points

- (1) One needs to work with a property that is stable under localization, which is fine here by **Lemma 9.0.2**.
- (2) Then, one can replace the original affine cover $\{\operatorname{Spec} A_i\}$ by principal open sets of $\operatorname{Spec} A_i$ that lie inside $\operatorname{Spec} B$. Here, $\operatorname{Spec} B$ is the open set of X , for which we want to show that B is Noetherian. This way *we may assume that X is affine*.
- (3) The next step is to assume that the affine cover is formed of principal open sets of both X and of the A_i . This way, *we may assume that the A_i are principal open sets of X* .
- (4) Then *we may assume that there are finitely many A_i , by X being affine*.
- (5) Then we are down to an algebra question that we solve by algebra. Here the question: $(f_1, \dots, f_r) = (1)$ as ideals of B , and B_{f_i} are Noetherian, then B is also Noetherian. This we suggest to solve differently than in Hartshorne, by using **Lemma 9.0.3**.

Lemma 9.0.3. *Let $f_1, \dots, f_r \in R$ be finitely many elements such that $(f_1, \dots, f_r) = (1)$. Then:*

- (1) $I = \bigcap_i (I : f_i^s)$ for every integer $s > 0$ and any ideal I of R ;
- (2) R_{f_i} is Noetherian for all i .

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Then, R is Noetherian.

Proof. (1): We only have to show $\supseteq$, as the other direction holds by definition of the colon ideals. Note that $\sqrt{(f_1^s, \dots, f_r^s)} = \sqrt{(f_1, \dots, f_r)}$, and hence $(f_1^s, \dots, f_r^s) = (1)$. With other words, we may assume that $s = 1$. In that case, by the assumption $(f_1, \dots, f_r) = (1)$, there exist elements $a_1, \dots, a_r \in R$ such that $1 = \sum a_i f_i$. Take then $x \in \bigcap_i (I : f_i)$. Then we have

$$x = \sum_i a_i \underbrace{f_i x}_{\uparrow} \in I$$

$x \in (I : f_i) \implies f_i x \in I$

(2): As for a fixed i we have $(I : f_i) \subseteq (I : f_i^2) \subseteq \dots$, and as point (1) holds for every s we obtain

$$I = \bigcap_i \left(\bigcup_{s>0} (I : f_i^s) \right) = \bigcap_i \phi_i^{-1} (R_{f_i} \cdot \phi_i(I))$$

(1) holds for every s , and for a fixed i we have $(I : f_i) \subseteq (I : f_i^2) \subseteq \dots$

$\phi_i : R \rightarrow R_{f_i}$ is the localization homomorphism

Now, if $I_1 \subseteq I_2 \subseteq \dots$ is an ascending chain of ideals of R , then chains $R_{f_i} \cdot \phi_i(I_j)$ formed of the extended ideals for a fixed i stabilize, as R_{f_i} is Noetherian. Then the above inequality concludes that the chain $I_1 \subseteq I_2 \subseteq \dots$ also stabilizes. $\square$

9.1 SINGULARITY NOTIONS

9.1.1 Regularity and integrality

Here we show.

Theorem 9.1.1. *If R is a regular Noetherian local ring, then R is a domain.*

Definition 9.1.2. If (R, m) is a regular Noetherian local ring, then $r_1, \dots, r_d \in m$ are called coordinates, if their images in m/m^2 are independent over R/m .

Lemma 9.1.3. *If (R, m) is a Noetherian local ring, and $r_1, \dots, r_d \in m$ are such that their images in m/m^2 generate m/m^2 as an R/m -vector space, then $m = (r_1, \dots, r_n)$.*

Proof. Apply Nakayama-lemma to $m/(r_1, \dots, r_n)$. We leave the details to the reader. $\square$

Lemma 9.1.4. *If (R, m) is a regular Noetherian local ring, and $r \in m$ is a coordinate, then $R/(r)$ is also a regular Noetherian local ring.*

Proof. Let n be the maximal ideal of $S = R/(r)$. As r is a coordinate, we have an exact sequence of $k = R/m = S/n$ vector-spaces (hw: work out the details):

$$0 \longrightarrow k \cdot r \longrightarrow m/m^2 \longrightarrow n/n^2 \longrightarrow 0$$

Hence, if $d = \dim R$, then it is enough to show that $\dim S = d - 1$.

$\dim S \leq d - 1$: We can extend r to a full set of coordinates $r = r_1, \dots, r_d$ of R . Let r'_i be the image of r_i in S . Then $(r'_2, \dots, r'_d) = n$ by Lemma 9.1.3. Hence, by Corollary 7.4.10, $\dim S \leq d - 1$.

$\dim S = d - 1$: Let $t'_1, \dots, t'_{\dim S}$ be elements of n such that $\sqrt{(t'_1, \dots, t'_{\dim S})} = n$. Let $t_i \in R$ be a lift of t'_i . Then $\sqrt{(r, t_1, \dots, t_{\dim S})} = m$. Hence, by Corollary 7.4.10 we have $\dim S \geq d - 1$. $\square$

By Proposition 8.5.18. and the primary decomposition (Theorem 8.5.21) of the “Rings and modules” notes we have

Lemma 9.1.5. *If $q_1, \dots, q_r$ are the associated primes of (0) in a Noetherian ring R , then*

$$\bigcup_{i=1}^r q_i = \{ r \in R \mid r \text{ is a zero divisor} \}$$

Lemma 9.1.6. *If R is a Noetherian local ring and $r \in R$ such that*

- (1) *r is not contained in any of the minimal primes of R , and*
- (2) *$R/(r)$ is a domain.*

Then, R is also a domain.

Proof. As $R/(r)$ is a domain, (r) is a prime ideal. Hence it contains some of the minimal primes q of R . Furthermore, by assumption (1), $q \subsetneq (r) = Rr$.

Take now $y \in q$ arbitrary. We may write $y = xr$ for $x \in R$. However, as $r \notin q$ and q is prime, we in fact have $x \in q$. This shows that we have $q = rq$. Then, Nakayama Lemma (Sheet 9, Exc 1 of “Rings and modules”) shows us that there is an element $z \in 1 + (r)$ such that $zq = 0$. However, then z is invertible, and hence $q = 0$. This implies that R is a domain (first R is reduced as the intersection of all primes is just (0) , and then since there is a single minimal prime, it is even a domain, as $\text{Spec } R$ is irreducible). □

The following lemma is the direct generalization of Lemma 7.4.9, by allowing two of the ideals to be non-primes. We phrase the statement in the contrapositive fashion, with respect to Lemma 7.4.9, as it is easier to prove it this way.

Lemma 9.1.7. PRIME AVOIDANCE II. *If $I_1, \dots, I_s, J$ are ideals such that*

- (1) *at most two of the I_j are not prime, and*
- (2) *$J \subseteq \bigcup_j I_j$,*

then $J \subseteq I_j$ for some $1 \leq j \leq s$.

Proof. We show the statement by induction on s . For $s = 1$, there is nothing to prove.

s = 2: Suppose that $J \not\subseteq I_j$ for both $j = 1, 2$. In the rest of the proof of case $s = 2$, let i always denote the other index with respect to j , that is, we have $\{i, j\} = \{1, 2\}$. Choose then $x_i \in J \setminus I_j$ for $j = 1, 2$. Then

$$x_i \in J \setminus I_j \subseteq (I_1 \cup I_2) \setminus I_j \subseteq I_i \tag{9.1.a}$$

In a similar fashion we have

$$x_1 + x_2 \in J \subseteq I_1 \cup I_2.$$

So, for one of the indices j we have $x_1 + x_2 \in I_j$. Let us fix now j to be this value, and let i be as above the other index. Then we have

$$x_i = \underbrace{(x_1 + x_2)}_{\substack{\uparrow \\ \in I_j}} - \underbrace{x_j}_{\substack{\uparrow \\ \in I_j \text{ by (9.1.a)}}} \in I_j.$$

This is a contradiction.

$s > 2$, the induction step: We may assume that I_n is a prime ideal. Also, by induction we may assume that for all $1 \leq j \leq s$ we have

$$J \not\subseteq \bigcup_{i \neq j} I_i$$

Thus we may pick

$$x_j \in J \setminus \left(\bigcup_{i \neq j} I_i \right) \subseteq I_j$$

Then

$$x = x_s + \prod_{i=1}^{s-1} x_i \in J \subseteq \bigcup_j I_j$$

Hence, we may fix an index j such that $x \in I_j$. There are two cases:

(1) If $j = s$, then

$$\prod_{i=1}^{s-1} x_i = x - x_s \in I_s$$

which contradicts the fact that $x_i \notin I_s$ for $i \neq s$ and that I_s is prime

(2) If $j \neq s$, then

$$x_s = x - \prod_{i=1}^{s-1} x_i \in I_j$$

which is again a contradiction.

□

Proof of Theorem 9.1.1. We prove it by induction on $\dim R$. Let m be the maximal ideal of R .

Case of $\dim R = 0$. As in this case $\frac{m}{m^2} = 0$, by Nakayama lemma we have $m = 0$. Therefore R is a field, and hence R is indeed a domain.

Induction step, $\dim R = n > 0$: Let $q_1, \dots, q_s$ be the minimal primes of R . As $\text{ht } m = \dim R > 0$, m is not contained in any of the q_i . Additionally, by Nakayama lemma $m^2 \neq m$ (see case $\dim R = 0$).

Hence, by the Prime avoidance lemma II (Lemma 9.1.7), we have

$$m \not\subseteq m^2 \cup \bigcup_i q_i.$$

Let $r \in m \setminus (m^2 \cup \bigcup_i q_i)$. By Lemma 9.1.4, $S = R/(r)$ is also a regular Noetherian local ring. Hence S is a domain by our induction hypothesis. Then, Lemma 9.1.6 concludes our proof.

□

Remark 9.1.8. What is even harder to prove, and hence we are not covering it in the present course, but it is useful to know about it (feel free to use them in exercises):

- (1) any localization of a regular Noetherian local ring is a regular Noetherian local ring, and
- (2) a regular local ring is a UFD, and hence it is integrally closed.

9.1.2 Singularity notions

We call the notions singularity notions that can be checked on local rings. So for example there will be an exercise on the exercise sheet saying that for a Noetherian ring R

- R is reduced if and only if R_p is reduced for all $p \in \operatorname{Spec} R$,
- R is integral if and only if R_p is integral,
- $\operatorname{Spec} R$ is irreducible if and only if $\operatorname{Spec} R_p$ is irreducible for all $p \in \operatorname{Spec} R$,
- if R is a domain, then R is integrally closed if and only if R_p is integrally closed for all $p \in R$,

As we explained in [Remark 9.1.8](#), the same statement holds for being regular, but it is much harder to prove it.

Remark 9.1.9. The situation is summarized in the following diagram:

$$\begin{array}{ccccccc}
 \text{all schemes} & & \supseteq & & \text{reduced schemes} & & \\
 \downarrow \cup & & & & \downarrow \cup & & \\
 \text{irreducible schemes} & \supseteq & \text{integral schemes} & \supseteq & \text{normal schemes} & \supseteq & \text{regular schemes}
 \end{array}$$

9.2 FURTHER PROPERTIES OF MORPHISMS

9.2.1 Finite type morphisms

Remark 9.2.1. Here an important point is that morphisms between schemes of finite type over k are of finite type. That is, morphisms between classical varieties are of finite type.

Example 9.2.2. Another very important set of examples are of the form $\operatorname{Spec} \left(\mathbb{Z}[x_1, \dots, x_n] / I \right) \rightarrow \operatorname{Spec} \mathbb{Z}$.

9.2.2 Finite morphisms

Here the important thing to realize is that we have learned in “Rings and modules” that if $B \hookrightarrow A$ is a ring homomorphism then A being a finite module over B is equivalent to A being integral over B (with other words an integral extension of B).

Example 9.2.3. Noether normalization gives a finite morphism.

Example 9.2.4. $\mathbb{A}_k^1 \setminus \{0\} \hookrightarrow \mathbb{A}_k^1$ is not finite as it corresponds to $k[x] \hookrightarrow k[x, x^{-1}]$.

Example 9.2.5. $k[x] \supseteq k[x^2] \cong k[t]$ yields a finite morphism $\mathbb{A}_x^1 \rightarrow \mathbb{A}_t^1$.

9.3 CLOSED SUBSCHEMES

The examples as in [Example 5.0.1](#), that is $\operatorname{Spec} \left(k[x, y] / (x^2y, xy^2) \right)$ yield closed subschemes of $\mathbb{A}_{x,y}^2$ supported on $V(xy)$. However, there is a special one out of all these subschemes, $\operatorname{Spec} \left(k[x, y] / (xy) \right)$, which is the *reduced induced closed subscheme* on $V(xy)$.

9.4 PRODUCTS

The idea:

- (1) tensor product yields products in the affine case
- (2) then we glue, where the cocycle condition is given by the unicity of the product.

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9.5 FIBERS OF MORPHISMS

Remark 9.5.1. Let $\lambda : B \rightarrow A$ be a ring homomorphism and $p \in \operatorname{Spec} B$. Let $f : X \rightarrow Y$ and $y \in Y$ be the corresponding morphism of schemes and the corresponding point of Y . Then we have

$$X \times_Y \operatorname{Spec} k(y) = \operatorname{Spec} \left((A \setminus \{0\})^{-1} (A/pA) \right)$$

In particular, if $p = m$ is maximal, then we have

$$X \times_Y \operatorname{Spec} k(y) = \operatorname{Spec} (A/mA)$$

Example 9.5.2. For f being the morphism of [Example 9.8.2](#) the fibers of f are isomorphic to three possible schemes:

- (1) $\operatorname{Spec}(k \oplus k)$
- (2) $\operatorname{Spec} L$ for a degree two field extension L of k
- (3) $\operatorname{Spec} k[x]/x^2$.

Remark 9.5.3. Surjective morphisms $f : X \rightarrow Y$ are many times thought of as *families* of their fibers.

For example, schemes over $\mathbb{Z}$ are often understood as families of their reductions mod p .

In fact, even more, schemes $X_{\mathbb{Q}}$ over $\mathbb{Q}$ are many times understood, by finding a scheme $X_{\mathbb{Z}}$ over $\mathbb{Z}$ such that

$$X_{\mathbb{Z}} \cong X_{\mathbb{Q}} \times_{\operatorname{Spec} \mathbb{Q}} \operatorname{Spec} \mathbb{Z}$$

In these situation $X_{\mathbb{Z}}$ is called a model of $X_{\mathbb{Q}}$ (over $\mathbb{Z}$), and the fiber of $X_{\mathbb{Z}} \rightarrow \operatorname{Spec} \mathbb{Z}$ over $p \in \operatorname{Spec} \mathbb{Z}$ is a reduction mod p of $X_{\mathbb{Q}}$.

Same notation is used for $\mathbb{Q}$ replaced by a number field K , and $\mathbb{Z}$ by the ring of integers R of K .

Remark 9.5.4. Reductions mod p of schemes over $\mathbb{Q}$ are not unique! For example $\mathbb{P}_{\mathbb{Z}}^1$ is a model of $\mathbb{P}_{\mathbb{Q}}^1$ over $\mathbb{Z}$, and hence $\mathbb{P}_{\mathbb{F}_p}^1$ is a reduction of $\mathbb{P}_{\mathbb{Q}}^1$ mod p .

However, note that the 2-nd veronese subring of $R_1 = \mathbb{Q}[x, y]$ (with the standard grading) is the ring $R_2 = \mathbb{Q}[u, v, w]/(uv - w^2)$, where $\deg u = \deg v = \deg w = 2$. Let R_3 be the same ring as R_2 , but with $\deg u = \deg v = \deg w = 1$. Finally, let $R_4 = \mathbb{Q}[u, v, w]/(2uv - w^2)$ with $\deg u = \deg v = \deg w = 1$. Then we have

$$\begin{array}{ccccccc} \mathbb{P}_{\mathbb{Q}}^1 = \operatorname{Proj} R_1 & \cong & \operatorname{Proj} R_2 & \cong & \operatorname{Proj} R_3 & \cong & \operatorname{Proj} R_4 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \boxed{\text{Proposition 8.2.1}} & & \boxed{\text{Multiplying the grading does not change Proj}} & & \boxed{\text{we may construct an isomorphism using Exercise II.2.14 from Hartshorne, by sending } u \mapsto 2u \text{ in one direction and } u \mapsto \frac{1}{2}u \text{ in the other direction}} \end{array}$$

Hence, $\mathcal{X} = \operatorname{Proj} \left(\mathbb{Z}[u, v, w]/(2uv - w^2) \right)$ is a model of $\mathbb{P}_{\mathbb{Q}}^1$ over $\mathbb{Z}$. In particular,

$$\mathcal{X}_2 = \operatorname{Proj} \left(\mathbb{Z}[u, v, w]/(2uv - w^2) \right) \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{F}_2 \cong \operatorname{Proj} \left(\mathbb{F}_2[u, v, w]/(2uv - w^2) \right) \cong \operatorname{Proj} \left(\mathbb{F}_2[u, v, w]/(w^2) \right)$$

By [Remark 9.5.5](#)

is also a reduction of $\mathbb{P}_{\mathbb{Q}}^1$ mod 2, which is not isomorphic to $\mathbb{P}_{\mathbb{F}_2}^1$. In fact it is a degree 2 infinitesimal thickening of $\mathbb{P}_{\mathbb{F}_2}^1$.

Remark 9.5.5. If S is a $\mathbb{N}$ -graded ring with $S_0 = A$, and B is a ring extension of A , then

$$\mathrm{Proj} S \times_{\mathrm{Spec} A} \mathrm{Spec} B = \mathrm{Proj} \left(S \underset{\uparrow}{\otimes_A} B \right)$$

The grading is given by the fact that $S = \bigoplus_{n \in \mathbb{N}} S_n$ is a direct sum of R -modules, which follows from the definition of a graded ring

To see this, construct a morphism

$$\mathrm{Proj} (S \otimes_A B) \rightarrow \mathrm{Proj} S \times_{\mathrm{Spec} A} \mathrm{Spec} B$$

by the universal property of products, then localize at homogeneous elements $f \in S$ to show that it is an isomorphism.

We note that the above products are called *base-changes*.

Remark 9.5.6. A particular base-change which is used plenty of times is the base-change to the generic point. When the base is $\mathrm{Spec} \mathbb{Z}$, then this is the “inverse” of taking a model. On the ring level both for Spec or for Proj this is given by localization at the multiplicative set $R \setminus \{0\}$, assuming that the base is $\mathrm{Spec} R$ for a domain R . The resulting scheme is called the *generic fiber* of the morphism.

A twist on this is the base-change to an algebraic closure L of the $\mathrm{Frac} R$. The resulting scheme is the *geometric generic fiber* of the morphism.

The general mantra is that if $\mathcal{P}$ is a singularity property (see [Section 9.1.2](#)), and $f : X \rightarrow Y$ is a morphism of schemes of finite type over $k = \bar{k}$ (the main example being morphisms of classical varieties), then there is a non-empty open set $U \subseteq Y$ such that

$$\text{the geometric generic satisfies } \mathcal{P} \iff \text{for every } k\text{-rational point } y \in U, X_y \text{ satisfies } \mathcal{P} \quad (9.5.a)$$

Note that k -rational points are the ones for which $k(y) \cong k$. We also note that fibers over such points in this case are called closed fibers. Additionally, it is particularly important to note that “geometric generic” cannot be replaced here by simply “generic”, see [Example 9.5.7](#).

Unfortunately, for each property $\mathcal{P}$, one needs to show (9.5.a) separately. For example, we will return to how (9.5.a) is show for $\mathcal{P}$ = “regular” in [Section 9.11.1](#). For $\mathcal{P}$ = “reduced” we will have an exercise showing one direction (geometric generic fiber is not reduced, then there is a non-empty open set of the base over which closed fibers are non-reduced). Because of the lack of time, we do not cover other directions/versions of (9.5.a).

Example 9.5.7. Let $k = \mathbb{F}_2$. $R = \left(k[x, y, z, u, v, w] / (x^2t + y^2u + z^2v) \right)$ with grading $\deg x = \deg y = \deg z = 1$ and $\deg u = \deg v = \deg w = 0$. Let $\mathcal{X} = \mathrm{Proj} R$. It is over $T = \mathrm{Spec} k[u, v, w]$.

Then $\mathcal{X}$ is reduced for example, because we can check that it is regular by the Jacobian condition. Consider the generic fiber $\mathcal{X}_\eta = \mathcal{X} \times_T (\mathrm{Spec} k(u, v, w))$. As the stalks of $\mathcal{O}_{\mathcal{X}_\eta}$ are all stalks of $\mathcal{O}_{\mathcal{X}}$ we obtain that $\mathcal{X}_\eta$ is also regular and hence reduced.

However, $\mathcal{X}_{\bar{\eta}} = \mathcal{X} \times_T (\mathrm{Spec} \overline{k(u, v, w)})$ is NOT reduced, and so are all the closed fibers.

With other words, the generic fiber of $\mathcal{X} \rightarrow T$ is a not geometrically reduced scheme (the opposite is characterized in exc II.3.15.b., and the above example explains why they are so important).

Examples of this type are particularly important, as in characteristic zero they do not appear, see Cor III.10.7 of Hartshorne, that we do not cover in this course because of the lack of time.

9.6 BASE-EXTENSIONS

Let X be a scheme of finite type over a perfect field k (e.g., k has characteristic 0, or k is finite). Let us understand how to think about the closed points of X in terms of $X_{\bar{k}} = X \times_{\text{Spec } k} \text{Spec } \bar{k}$. For that we may restrict to an affine open set $\text{Spec } A$, which by the finite type assumption is the quotient of $k[x_1, \dots, x_n]$. With other words, we may assume that $X = \text{Spec } A$, or equivalently we may think of X being a closed subscheme of $\mathbb{A}_k^n$. Take a maximal ideal $m \subseteq A$. In “Rings and modules” (Thm 6.1.11 of the “Rings and modules” notes), we showed that if a finitely generated k algebra is a field then it is an algebraic extension. Putting this together with the assumption that it is a finitely generated k -algebra, we obtain that it is a finite extension. Hence $[L = A/m : k] < \infty$. This yields a bijection

$$\boxed{\text{closed points of } X} \longleftrightarrow \boxed{k\text{-algebra homomorphisms } \phi : A \rightarrow L, \text{ where } [L : k] < \infty}$$

Note that $\phi : A \rightarrow L$ induces a $\bar{k}$ -algebra homomorphism

$$A \otimes_k \bar{k} \xrightarrow{\phi \otimes_k \bar{k}} L \otimes_k \bar{k}$$

The closed points of $X_{\bar{k}}$ lying over the closed point ϕ of X , are exactly the surjective $\bar{k}$ -algebra homomorphisms $A \rightarrow \bar{k}$ given composing $\phi \otimes_k \bar{k}$ with an $\bar{k}$ -algebra homomorphism $L \otimes_k \bar{k} \rightarrow \bar{k}$ (which is necessarily surjective, as the target is $\bar{k}$ itself). This follows from the universal property of tensor product, and the following commutative diagram

$$\begin{array}{ccccc} & & \xrightarrow{\phi \otimes_k \bar{k}} & & \\ & \nearrow & & \searrow \exists! & \\ A \otimes_k \bar{k} & \xrightarrow{\phi \otimes_k \bar{k}} & L \otimes_k \bar{k} & \dashrightarrow & \bar{k} \\ \uparrow & & \uparrow & & \uparrow \\ A & \xrightarrow{\phi} & L & & \bar{k} \end{array}$$

Lemma 9.6.1. *We have*

$$|\{ \bar{k}\text{-algebra homomorphisms } L \otimes_k \bar{k} \rightarrow \bar{k} \}| = [L : k]$$

Moreover, such $\bar{k}$ -algebra homomorphisms form a $\text{Gal}_k(\bar{k})$ -orbit of $\bar{k}$ -algebra homomorphisms $L \otimes_k \bar{k} \rightarrow \bar{k}$ (we act by post-composing with the Galois elements). Moreover, the stabiliser of an element of the orbit is $\text{Gal}_L(\bar{k})$.

Proof. In this proof we use a lot of statements from Galois theory. They all can be found in the “anneaux et corps” notes. Let us know, if you have problems finding them.

First, note that the left side is in bijection with the k -algebra homomorphism $\phi : L \rightarrow \bar{k}$. Indeed, $\phi \otimes_k \bar{k}$ yields a $\bar{k}$ -algebra homomorphism $L \otimes_k \bar{k} \rightarrow \bar{k}$. And, in the other direction, a $\bar{k}$ -algebra homomorphism yields a k -algebra homomorphism by taking $\phi = \xi|_{L \otimes_k \{1\}}$. We leave it to the reader to check that these are inverses of each-other. In any case, we obtained that it is enough to understand k -algebra homomorphisms $\phi : L \rightarrow \bar{k}$.

As $k \subseteq L$ is a finite separable extensions (remember that we assumed that k is perfect), it is a simple extension by $\alpha \in \bar{k}$ with minimal polynomial $m_{\alpha,k}$ over k of degree $d = [L : k]$. Let $\alpha = \alpha_1, \dots, \alpha_d$ be the roots of $m_{\alpha,k}$ (by separability there are no multiple roots). As ϕ is a k -algebra homomorphism and the coefficients of $m_{\alpha,k}$ are in k , we have $\phi(\alpha) = \alpha_i$ for some i . So there are at most d options for ϕ .

We show that all d -options are realized. As $\bar{k} \supseteq k(\alpha_i) \cong k[x]/m_{\alpha,k}$, there is a unique isomorphism $\phi_i : L \rightarrow k(\alpha_i)$ such that $\phi_i(\alpha) = \alpha_i$. As $k(\alpha_i)$ is a subfield of $\bar{k}$, this yields d different homomorphisms $L \rightarrow \bar{k}$.

The Galois statement follows from the fact that α_i are the elements in the Galois orbit of α and α_i are fixed by the action of $\text{Gal}_L(\bar{k})$. $\square$

Putting everything together we obtain that there are $[L : k]$ closed points of $X_{\bar{k}} = X \times_{\text{Spec } k} \text{Spec } \bar{k} = \text{Spec } (A \otimes_k \bar{k})$ lying over a fixed closed point $A \rightarrow L$ of X . Additionally these form a Galois orbit for $\text{Gal}_k(\bar{k})$, with stabiliser equal to $\text{Gal}_k(L)$.

Furthermore, if we have a closed point $\xi : A \otimes_k \bar{k} \rightarrow \bar{k}$ of $X_{\bar{k}}$, then $\xi|_{A \otimes \{1\}} : A \rightarrow \bar{k}$ is a k -algebra homomorphism. As A is a finitely generated k -algebra, so is the image of $\xi|_{A \otimes \{1\}}$. Hence it is contained in a finite field extension $k \subseteq L \subseteq \bar{k}$. This shows that all Galois orbits of closed points of $X_{\bar{k}}$ lie over a closed point of X .

If we think about the closed points of $X_{\bar{k}}$ as points of the vector space $\bar{k}^n$, then we obtain:

$$\boxed{\text{closed points } x \text{ of } X} \leftrightarrow \boxed{\text{Gal}_k(\bar{k})\text{-orbits of closed points } X_{\bar{k}}, \text{ with stabiliser } \text{Gal}_k(k(x)) \text{ inside } \mathbb{A}_{\bar{k}}^n}$$

Additionally we have that the degree $\deg L$ of the closed point $A \rightarrow L$ of X equals the size of the Galois orbit.

Example 9.6.2. Consider $X = \text{Spec } \left(\mathbb{Q}[x, y] / (x^3 + y^3 + 1) \right)$. Fermat's last theorem (showed by Wiles) says that the only $\mathbb{Q}$ -rational points are $(-1, 0)$ and $(0, -1)$. To understand the other points of X , we have to use the above equivalence. For example:

- (1) $(-2, \sqrt[3]{7})$ is a closed point of $X_{\bar{\mathbb{Q}}}$, and as $\sqrt[3]{7}$ has a Galois orbit of size 3:

$$\sqrt[3]{7}, \quad e^{\frac{2\pi i}{3}} \sqrt[3]{7}, \quad e^{\frac{4\pi i}{3}} \sqrt[3]{7},$$

we obtain a closed point $\text{Spec } k(x) \rightarrow X$ of degree 3 corresponding to the maximal ideal $(x + 2, y^3 - 7)$.

- (2) $(i, \sqrt[3]{i-1})$ is a closed point of $X_{\bar{\mathbb{Q}}}$. Here note that $x^3 - (i-1)$ is irreducible over $\mathbb{Q}(i)$ as it has degree 3 and it has no root in $\mathbb{Q}(i)$. To see that it has no root, note that $N(i-1) = 2$, and hence a root w would have $N(w) = \sqrt[3]{2}$ which is impossible (the norm $N(z\bar{z})$ is always rational for an element of $\mathbb{Q}(i)$). Hence, it follows that $[\mathbb{Q}(i, \sqrt[3]{i-1}) : \mathbb{Q}] = 6$, and hence the Galois orbit has 6 elements. If we put the first coordinate into the picture as well, the Galois orbit has still 6 elements and it is

$$\begin{aligned} & \left(i, \sqrt[3]{i-1} \right), \quad \left(i, e^{\frac{2\pi i}{3}} \sqrt[3]{i-1} \right), \quad \left(i, e^{\frac{4\pi i}{3}} \sqrt[3]{i-1} \right) \\ & \left(-i, \sqrt[3]{-i-1} \right), \quad \left(-i, e^{\frac{2\pi i}{3}} \sqrt[3]{-i-1} \right), \quad \left(-i, e^{\frac{4\pi i}{3}} \sqrt[3]{-i-1} \right) \end{aligned}$$

We obtain a closed point $\text{Spec } k(x) \rightarrow X$ of degree 6, corresponding to maximal ideal $(x^2 + 1, y^3 - x + 1)$.

- (3) etc.

9.7 NORMALITY

9.7.1 Integral elements

Recall that if $R \subseteq S$ is a ring extension, then $s \in S$ is integral over R if it satisfies a monic polynomial (so one with leading coefficient 1) with coefficients in R .

Recall further that if M is an R -module, then the annihilator of M is by definition the following ideal of R : $\text{Ann}_R(M) := \{ r \in R \mid \forall m \in M : r \cdot m = 0 \}$.

Proposition 9.7.1. *For a ring extension $R \subseteq S$, $s \in S$ is integral over R if and only if there is a faithful module M over $R[s]$ such that M is finitely generated as an R -module. (Here faithful means that $\text{Ann}_{R[s]}(M) = 0$.)*

Proof. $\Rightarrow$ The faithful module will be $R[s]$ itself. It is faithful because no non-zero element of $R[s]$ can kill $1 \in R[s]$. It is finitely generated as an R -module because s satisfies the relation $s^n = \sum_{i=0}^{n-1} r_i s^i$ for some integer $n > 0$ and elements $r_i \in R$. Thus, $R[s]$ is generated as an R -module by $\{1, s, \dots, s^{n-1}\}$.

$\Leftarrow$ For this direction we just need to repeat the proof of the Nakayama lemma: let $m_1, \dots, m_r$ be generators of M as an R -module and consider the action of s on M . This action can be described by a matrix (a_{ij}) , with entries in R . That is, we may find (non-unique!) $a_{ij} \in R$ such that

$$sm_i = \sum_{j=1}^r a_{ij} m_j.$$

Then the matrix $A = (\delta_{ij}s - a_{ij})_{i,j=1,\dots,r}$ with entries in $R[s]$ takes the vector $[m_1, \dots, m_r]^T$ to zero (here δ_{ij} is zero if $i \neq j$ and 1 if $i = j$). In particular so does $A^*A = (\det A)\text{Id}_M$, where A^* is the adjugate matrix. However $\det A$ is just an element of $R[s]$. So, we obtain that multiplying any m_i with this element of $R[s]$ is zero. As m_i is a generator set of M as an R -module, it follows that $(\det A) \cdot M = 0$. Using now the faithfulness assumption, we obtain that $\det A = 0$. However, computing $\det A$ yields a monic polynomial of s over R . As this is zero, we obtain that s is integral over R . $\square$

Definition 9.7.2. Let $R \rightarrow S$ be a ring extension. We say R is *integrally closed in S* if for any $s \in S$ integral over R , then $s \in R$.

Let R be a domain. We say it is *integrally closed* if it is integrally closed in $\text{Frac}(R)$. If R is integrally closed, we say that $\text{Spec}(R)$ is *normal*.

We recall

Proposition 9.7.3. Let R be a UFD. Then R is integrally closed. In particular, PID are integrally closed.

Proof. Let $K := \text{Frac}(R)$ and let $x \in K$ which satisfies the monic equation

$$x^n + \sum_{i=1}^{n-1} a_i x^i = 0.$$

We can write $x = \frac{s}{t}$, where $\gcd(s, t) \in R^*$ by UFD hypothesis. Then we have the equation

$$s^n + t \left(\sum_{i=0}^{n-1} a_i s^i t^{n-1-i} \right) = 0.$$

In particular t divides s^n . Therefore $\gcd(s, t) = \gcd(s^n, t) = t \in R^*$. Therefore $x = \frac{s}{t} \in R^*$. $\square$

Example 9.7.4. The ring $k[t]$ is integrally closed (so we say that $\mathbb{A}_t^1$ is a normal variety) as $k[t]$ is a UFD. Also $k[x_1, \dots, x_n]$ is UFD and thus integrally closed.

The ring $R = k[x, y]/(x^3 - y^2)$ is not integrally closed, as $\frac{y}{x} \in \text{Frac}(R)$ satisfies the monic polynomial $P(t) = t^3 - y$. Note that $\text{Spec}(R)$ is a cuspidal curve in $\mathbb{A}^2$.

These examples suggests that normality (and its failure) captures some singularity notions.

Example 9.7.5. It's not difficult to construct examples of integrally closed domains which are not a UFD.

Consider $R = k[x, y, z]/(z^2 - xy)$ where $\text{char}(k) \neq 2$. This is not a UFD as

$$z^2 = z \cdot z = x \cdot y.$$

We now show it is integrally closed. Note that R is an integral extension of $k[x, y]$ (apply [Proposition 9.7.1](#)). Let $\alpha = u + vz \in \text{Frac}(R)$ be an integral element over R for some elements

$u, v \in k(x, y)$. We now show that u and v belong to $k[x, y]$. As R is integral over $k[x, y]$ we deduce that α is integral over R and let $P \in k[x, y][T]$ such that $P(\alpha) = 0$. Consider the minimal polynomial Q of α over $k(x, y)$:

$$T^2 - 2uT + u^2 - xyv^2$$

As Q is the minimal polynomial of α , it must divide $P \in k[x, y][T]$ and so we deduce that $Q \in k[x, y][T]$.

In particular, $2u$ and $u^2 - xyv^2$ belong to $k[x, y]$. As $p \neq 2$, then $u \in k[x, y]$. Then $xyv^2 \in k[x, y]$. Write $v = \frac{R}{S}$ where R and S are coprime. We have $S^2 \mid (xyR^2)$ and therefore $S^2 \mid xy$, which is impossible unless S is a constant, showing that $v \in k[x, y]$.

9.7.2 Local Noetherian integrally closed domains of dimension 1 are discrete valuation rings

Proposition 9.7.6. *If (R, m) is a local, Noetherian domain of dimension 1, then the following holds.*

(1) *For each non-zero ideal $I \subseteq R$, $\sqrt{I} = m$.*

If R is integrally closed, then

(2) *The ideal m is principal, that is, there exists $t \in R$ such that $m = (t)$.*

Proof. (1) First we describe the prime ideals of R . As it is a domain, (0) is prime. Thus, as the dimension is 1, any non-zero prime is maximal (as it contains (0)). So the only prime ideals are m and (0) .

As $\sqrt{I}$ is radical, according to [Proposition 7.3.2](#), it can be written in a unique way as an intersection of prime ideals. But as I is non-zero, $\sqrt{I}$ is the intersection of the only non-zero prime, m . Thus $\sqrt{I} = m$.

(2) Fix $0 \neq a \in m$. If $(a) = m$ we are ready, so we may assume that $(a) \subsetneq m$.

Claim. *For some integer $i \geq 1$, the containment $(a) \supseteq m^i$ holds.*

Proof. We have $\sqrt{a} = m$ by point [Proposition 9.7.6](#), and so it is enough to show that given some ideal I , $\sqrt{I}^i \subseteq I$ for $i \gg 0$. Let $a_1, \dots, a_l$ be generators of $\sqrt{I}$. Since they are in the radical, there is some integer $r > 0$ such that for each i , $a_i^r \in I$. It follows that $(\sqrt{I})^{1+l(r-1)} \subseteq I$, by the pigeonhole principle. This finishes the proof of the claim. $\square$

Fix i to be the minimal as above. As we assumed that $(a) \subsetneq m$, $i > 1$. In particular, for this choice, $i-1 \geq 1$, and furthermore we may choose $b \in m^{i-1} \setminus (a)$. Then, $bm \subseteq m^i \subseteq (a)$, and therefore $\frac{b}{a}m \subseteq R$.

Claim. *We have $\frac{b}{a} \cdot m \not\subseteq m$.*

Proof. Assume that $\frac{b}{a} \cdot m \subseteq m$. Then, m is a finitely generated R -module which is a faithful $R[\frac{b}{a}]$ -module. Hence, by [Proposition 9.7.1](#), $\frac{b}{a}$ is integral over R . As R is integrally closed this means that $\frac{b}{a} \in R$, which contradicts our assumption that $b \notin (a)$. $\square$

The above claim implies that $\frac{b}{a}m$ is an ideal of R , which is not contained in the maximal ideal. Hence, $\frac{b}{a}m = R$, and hence there is a $t \in m$ such that $\frac{b}{a}t = 1 \Leftrightarrow a = bt$. In particular, $m = \frac{a}{b}R = tR$. □

Definition 9.7.7. The element t of the above proposition is called a *local parameter* if R .

Remark 9.7.8. By the above remark, local parameters are defined up to multiplication by a unit.

Definition 9.7.9. If K is a field, a *discrete valuation* of K is a map $\nu : K \setminus \{0\} \rightarrow \mathbb{Z}$ such that

- (1) $\nu(ab) = \nu(a) + \nu(b)$, and
- (2) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$.

The *valuation ring* of ν is then $\{f \in K \mid \nu(f) \geq 0\}$. (It follows from the above two axioms on ν that $\{f \in K \mid \nu(f) \geq 0\}$ is closed under multiplication and addition.)

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Example 9.7.10. For a prime number p and for the field $\mathbb{Q}$, $\nu_p(a/b) = \nu_p(a) - \nu_p(b)$ gives a valuation, where ν_p is the p -adic valuation counting the number of times p appears as a prime factor. Then the valuation ring is $\mathbb{Z}_{(p)} \subseteq \mathbb{Q}$.

Similarly $k(t) \setminus \{0\} \ni \frac{f(t)}{g(t)} \mapsto \nu_t(f(t)) - \nu_t(g(t))$ gives a discrete valuation, where ν_t counts how many times t is an irreducible factor.

Corollary 9.7.11. Let (R, m) be a local, Noetherian domain of dimension 1 such that R is either integrally closed, or regular. Set $L = R/m$. Then

- (1) $\dim_L m^i / m^{i+1} = 1$ (and so even if we assume that R is an integral domain, we obtain that it is in fact regular),
- (2) every element $r \in R$ can be written as ut^a , where u is a unit, and $\nu_R(r) := a$ is a uniquely determined integer, depending only on r and R .
- (3) R is the valuation ring for the discrete valuation on $\text{Frac}(R)$ given by

$$\text{Frac}(R) \setminus \{0\} \ni \frac{f}{g} \mapsto \nu_R(f) - \nu_R(g) =: \nu_R\left(\frac{f}{g}\right).$$

Proof. We note that for either assumption (i.e., R is integrally closed and R is regular), then we have $m = Rt$ for some $t \in R$. For the integrally closed case this is shown in [Proposition 9.7.6](#). In the regular case, this is shown in [Lemma 9.1.3](#). This is the only fact that we will use about R .

- (1) According to [Proposition 9.7.6](#), $\dim_L (m^i / m^{i+1}) = \dim_L ((t^i) / (t^{i+1}))$.

This yields a L -linear surjection $R/(t) \twoheadrightarrow (t^i)/(t^{i+1})$ defined by $[1] \mapsto t^i$. We claim that this surjection is a bijection, which will conclude our statement. As $\dim_L R/(t) = 1$, for this it is enough to show that $(t^i)/(t^{i+1}) \neq 0$ or equivalently that $(t^{i+1}) \subsetneq (t^i)$. Assume the contrary. Then, $t^i = rt^{i+1}$ for some $r \in R$. As R is a domain, this implies that $1 = rt$. Hence, t is a unit, and hence $m = R$. This is impossible, as the maximal ideal of a 1-dimensional local ring has to be a non-trivial ideal.

- (2) First we note that $\bigcap_{j \geq 1} (t^j) = \{0\}$. Indeed, if $0 \neq f \in \bigcap_{j \geq 1} (t^j)$, then the ideals $(f/t) \subsetneq (f/t^2) \subsetneq \dots$ form an infinite increasing sequence of ideals, contradicting Noetherianity.

Then, we may set j to be the largest integer such that $r \in (t^j)$. In particular, $r = ut^j$, where $u \notin (t)$. However, then u is not in the unique maximal ideal of our local ring R , which implies that it is a unit (otherwise there would be a maximal ideal containing it).

Lastly, the unicity of a as in the statement follows, since a can be characterized as the largest integer such that $r \in m^a$.

- (3) First we show that the above formula indeed gives a valuation. For that fix $r_i = u_i t^{a_i}$ for $i = 1, 2, 3, 4$ such that u_i is a unit. Then:

- (i) WELL-DEFINEDNESS: if $r_1 r_4 = r_2 r_3$, then $u_1 u_4 t^{a_1+a_4} = u_2 u_3 t^{a_2+a_3}$, where $u_1 u_4 = u_2 u_3$ is a unit. In particular by the unicity of the power of t , $a_1 + a_4 = a_2 + a_3$. Equivalently, $a_1 - a_2 = a_3 - a_4$, which is exactly saying that if $\frac{r_1}{r_2} = \frac{r_3}{r_4}$, then

$$\nu_R \left(\frac{r_1}{r_2} \right) = \nu_R(r_1) - \nu_R(r_2) = \nu_R(r_3) - \nu_R(r_4) = \nu \left(\frac{r_3}{r_4} \right).$$

- (ii) MULTIPLICATIVE PROPERTY: We have:

$$\frac{r_1}{r_2} \frac{r_3}{r_4} = \frac{u_1 t^{a_1}}{u_2 t^{a_2}} \frac{u_3 t^{a_3}}{u_4 t^{a_4}} = \frac{u_1 u_3 t^{a_1+a_3}}{u_2 u_4 t^{a_2+a_4}},$$

where $u_1 u_3$ and $u_2 u_4$ are units. So, we obtain that

$$\nu_R \left(\frac{r_1}{r_2} \frac{r_3}{r_4} \right) = \nu_R(r_1) + \nu_R(r_3) - \nu_R(r_2) - \nu_R(r_4) = \nu_R \left(\frac{r_1}{r_2} \right) + \nu_R \left(\frac{r_3}{r_4} \right).$$

- (iii) ADDITIVE PROPERTY: assuming that $a_1 - a_2 \leq a_3 - a_4$, which we may assume by symmetry, the following holds:

$$\begin{aligned} \frac{r_1}{r_2} + \frac{r_3}{r_4} &= \frac{u_1 t^{a_1}}{u_2 t^{a_2}} + \frac{u_3 t^{a_3}}{u_4 t^{a_4}} = \frac{u_1 u_4 t^{a_1+a_4} + u_2 u_3 t^{a_2+a_3}}{u_2 u_4 t^{a_2+a_4}} \\ &= \frac{(u_1 u_4 + u_2 u_3 t^{a_2+a_3-a_1-a_4}) t^{a_1+a_4}}{u_2 u_4 t^{a_2+a_4}} = \frac{(u_1 u_4 + u_2 u_3 t^{a_2+a_3-a_1-a_4}) t^{a_1}}{u_2 u_4 t^{a_2}}. \end{aligned}$$

The assumption $0 \leq a_2 + a_3 - a_1 - a_4$ implies that $u_1 u_4 + u_2 u_3 t^{a_2+a_3-a_1-a_4} \in R$. So, the above computation shows that

$$\begin{aligned} \nu_R \left(\frac{r_1}{r_2} + \frac{r_3}{r_4} \right) &\geq \underbrace{\nu_R(r_1) - \nu_R(r_2)}_{\text{we assumed that } a_1 - a_2 \leq a_3 - a_4} = \min\{\nu_R(r_1) - \nu_R(r_2), \nu_R(r_3) - \nu_R(r_4)\} \\ &= \min \left\{ \nu_R \left(\frac{r_1}{r_2} \right), \nu_R \left(\frac{r_3}{r_4} \right) \right\}. \end{aligned}$$

Now, we prove that R is the valuation ring for ν_R . So, take $f \in K := \text{Frac}(R)$, such that $\nu_R(f) \geq 0$. We may write $f = r_1/r_2$ for some $r_i \in R$, for which we have the presentation $r_i = u_i t^{a_i}$ as above. Then by definition $a_1 - a_2 \geq 0$. So, $f = \frac{r_1}{r_2} = \frac{u_1}{u_2} t^{a_1-a_2}$, which is in R , as u_2 is a unit, and $a_1 - a_2 \geq 0$.

□

9.7.3 Discrete valuation rings are local Noetherian integrally closed domains of dimension 1

Recall that the valuation ring associated to a discrete valuation $\nu: K^* \rightarrow \mathbb{Z}$ of a field K is defined by

$$\mathcal{O}_\nu := \{\xi \in K : \xi = 0 \text{ or } \nu(\xi) \geq 0\}.$$

Proposition 9.7.12 (Discrete valuation rings). *Let K be a field, and $\nu: K^* \rightarrow \mathbb{Z}$ a non-trivial valuation. Then $\mathcal{O}_\nu$ is a one-dimensional Noetherian local ring with maximal ideal*

$$\mathfrak{m}_\nu = \{\xi \in K : \xi = 0 \text{ or } \nu(\xi) > 0\},$$

which is principal. Additionally, $\mathcal{O}_\nu$ is a PID, is integrally closed, and its ideals are totally ordered by inclusion.

Proof. (1) $\mathcal{O}_\nu$ is indeed a ring. The properties of a valuation are tailored to this end. Note that

$$\nu(1) = \nu(1^2) = 2\nu(1)$$

so $\nu(1) = 0$.

(2) Given $x \in K^*$, either $x \in \mathcal{O}_\nu$ or $x^{-1} \in \mathcal{O}_\nu$. Indeed

$$0 = \nu(xx^{-1}) = \nu(x) + \nu(x^{-1})$$

so $\nu(x^{-1}) = -\nu(x)$. Our claim easily follows.

(3) We may assume that ν is surjective. Indeed, we have seen in the previous point that

$$\nu: (K^*, \cdot) \longrightarrow (\mathbb{Z}, +)$$

is a group morphism. Since it is not trivial, its image is a non-trivial subgroup of $\mathbb{Z}$, and any such subgroup is isomorphic to $\mathbb{Z}$.

(4) $\mathcal{O}_\nu^* = \{\xi \in K^* : \nu(\xi) = 0\}$. This is clear from the second claim.

(5) $\mathfrak{m}_\nu$ is the unique maximal ideal. From the properties of a valuation, it is easily seen that $\mathfrak{m}_\nu$ is an ideal. It is the unique maximal one because $\mathcal{O}_\nu - \mathfrak{m}_\nu = \mathcal{O}_\nu^*$.

(6) $\mathfrak{m}_\nu$ is principal. Since ν is surjective, there is some element π of $\mathcal{O}_\nu$ such that $\nu(\pi) = 1$. We claim that $\mathfrak{m}_\nu = (\pi)$. Pick $s \in \mathfrak{m}_\nu$; then

$$\nu\left(\frac{s}{\pi^{\nu(s)}}\right) = 0.$$

In particular $\frac{s}{\pi^{\nu(s)}} \in \mathcal{O}_\nu^*$ and so $s \in (\pi)$. Our claim follows.

(7) For every element $s \in \mathcal{O}_\nu$, there is $u \in \mathcal{O}_\nu^*$ such that $s = u\pi^{\nu(s)}$. Follows from the proof of the previous claim.

(8) $\mathcal{O}_\nu$ is a PID. Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_\nu$. Let $r = \min_n \{\pi^n \in \mathfrak{a}\}$. It exists by the previous claim and the fact that $\mathbb{N}$ is well-ordered. Then $\mathfrak{a} = (\pi^r)$.

(9) The ideals of $\mathcal{O}_\nu$ are totally ordered by the inclusion. Indeed, the chain

$$\mathfrak{m}_\nu = (\pi) \supset (\pi^2) \supset (\pi^3) \supset \cdots \supset 0$$

exhausts all the ideals.

(10) $\mathcal{O}_\nu$ is 1-dimensional. The only prime ideals are (π) and 0.

- (11) $\mathcal{O}_\nu$ is Noetherian. It is apparent from claim (9).
- (12) $\mathcal{O}_\nu$ is integrally closed. Suppose that $x \in K$ is integral over $\mathcal{O}_\nu$, and assume that $x \notin \mathcal{O}_\nu$. Then $x^{-1} \in \mathfrak{m}_\nu$ as observed above. Let

$$x^n + \sum_{i=0}^{n-1} a_i x^i = 0$$

be an equation for x with $a_i \in \mathcal{O}_\nu$. Multiplying by x^{-n} we get:

$$1 + \sum_{i=0}^{n-1} a_i x^{i-n} = 0$$

and thus $1 \in \mathfrak{m}_\nu$, contradiction. (Alternatively, a PID is integrally closed by **Proposition 9.7.3** so we could conclude from claim (8).)

□

9.7.4 Wrapping up DVRs

To sum it up:

Theorem 9.7.13. *Let R be a Noetherian local ring. Then the following are equivalent:*

- (1) R is a integrally closed domain of dimension 1,
- (2) R is regular of dimension 1, and
- (3) R is a discrete valuation ring (DVR for short).

9.7.5 Examples

Example 9.7.14. $X = \text{Spec} \left(k[x, y] / (xy) \right)$ is not normal. In particular, if $p = (x, y)$ is the origin, then $\mathcal{O}_{X,p} = k[x, y]_p / (xy)$ is not integrally closed. Indeed, if m is the maximal ideal, then $\dim_k m/m^2 = 2$, and so $\mathcal{O}_{X,p}$ is not DVR.

Example 9.7.15. On the other hand if X is a normal scheme of dimension 1, then $\mathcal{O}_{X,x}$ is regular.

Definition 9.7.16. A scheme C is called a *regular curve* if it is a Noetherian regular scheme of dimension one. Equivalently, it is a Noetherian integral scheme of dimension such that for all closed points $c \in C$, the local ring $\mathcal{O}_{C,c}$ is a DVR.

9.7.6 What we do not cover from normality

We do not cover the statement that if R is a finitely generated k -algebra which is a domain, then so is its integral closure in L , where L is a finite field extension of $\text{Frac}(R)$.

That is a the normalization morphism $X^\nu \rightarrow X$ is finite for a scheme X of finite type over k .

Feel free to use this statement though.

9.8 FINITE MORPHISMS

In Rings and modules we showed that if $f : X \rightarrow Y$ is a dominant finite morphism, then $\dim X = \dim Y$ (it follows directly from the algebra statement of Corollary 7.4.4 of the “Rings and modules” notes).

Definition 9.8.1. If $m < n$ are integers, then a projection $\mathbb{P}^n \supseteq U \rightarrow \mathbb{P}^m$ is the morphism defined by the embedding of graded rings $k[x_0, \dots, x_m] \hookrightarrow k[x_0, \dots, x_n]$ (we use standard grading) and by Exercise II.2.14, and possibly by applying a coordinate transformation on both ends (so a morphism given by linear transformation of the coordinates). We call $V(x_{m+1}, \dots, x_n)$ the indeterminacy locus of the projection, because $U = \mathbb{P}^n \setminus V(x_{m+1}, \dots, x_n)$.

By general theory, if $X \subseteq \mathbb{P}_k^2$ is a projective curve, $P \in \mathbb{P}_k^2 \setminus X$ is a closed point, and $\pi : U = \mathbb{P}_k^2 \setminus \{P\} \rightarrow \mathbb{P}^1$ is a projection, then $\pi|_X : X \rightarrow \mathbb{P}^1$ is a finite morphism. This morphism is used crucially in the theory of curves to understand X . We are not able to cover in the present course that $\pi|_X$ is finite, nevertheless, in concrete situations we already have the tools to verify it.

Before proceeding to such concrete situations, let us note that the above construction works also if $P \in X$ if X is regular. That is, the induced morphism $X \setminus \{P\} \rightarrow \mathbb{P}^1$ extends to a unique finite morphism $X \rightarrow \mathbb{P}^1$. If we proceed in a good pace with the material, we might be cover at least this extension property, but not the fact that it is finite.

In any case, let us consider some actual example of the above:

Example 9.8.2. Let k be algebraically closed and for some $a, b \in k$ let

$$X = V(y^2 - (x^3 + axz^2 + bz^3)) \subseteq \mathbb{P}_{x,y,z/k}^2$$

We leave it as an exercise that if $\text{char}(k) \neq 2, 3$, then X is regular if and only if $4a^3 + 27b^2 \neq 0$ (apply Proposition 8.3.4 on affine charts). Let us assume then that $4a^3 + 27b^2 \neq 0$.

Note that $[0, 1, 0] \in \mathbb{P}_k^2$ is actually contained in X , nevertheless we consider projection from $P = [0, 1, 0]$. That is, we do the projection induced by $k[x, z] \hookrightarrow k[x, y, z]$. Then the induced morphism $X \rightarrow \mathbb{P}^1$ is the unique extension (we will learn this extension property later) of the map $X \setminus \{P\} \rightarrow \mathbb{P}^1$ induced by the graded homomorphism of rings:

$$\phi : k[x, z] \rightarrow k[x, y, z] / (zy^2 - (x^3 + axz^2 + bz^3))$$

Note that $zy^2 - (x^3 + axz^2 + bz^3)$ is a homogeneous element of $k[x, y, z]$, so we are quotienting out by a homogeneous element, and hence the quotient ring is a graded ring.

Exercise: what does this morphism look like using the classical language?

In any case, using scheme theory, it is easy to see how this morphism looks like on affine charts, we just have localize and pass to the degree 0 part by Proposition II.2.5 of Hartshorne (we are secretly using some functoriality here that we let you think through):

$$(\phi_z)_0 : R = k[x, z, z^{-1}]_0 = k[x] \rightarrow \left(k[x, y, z, z^{-1}] / (zy^2 - (x^3 + axz^2 + bz^3)) \right)_0 = k[x, y] / (y^2 + (x^3 - ax + b))$$

Exercise: do the same passage to affine open set using the classical language.

Above we see that $R \subseteq S$ is an integral extension (as $y \in S$ satisfies a the monic equation $(y^2 + (x^3 - ax + b))$), and hence we obtain that $X \setminus \{P\} \rightarrow \mathbb{P}^1$ is finite at least over $\mathbb{A}^1 \cong D(z) \subseteq \mathbb{P}^1$.

This is one of the most frequently looked finite morphism of affine curves. The curve X is what we call an elliptic curve.

Definition 9.8.3. Let $f : X \rightarrow Y$ be a finite morphism of normal schemes and let $x \in X$ and $y \in Y$ be codimension one points such that $f(x) = y$. Then there is an induced ring homomorphism $f^{\#,x} : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ with local parameters t_y and t_x (local parameter=generator of the maximal ideal). Note that local parameters exist and $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}$ are normal local rings of dimension 1, and hence they are DVRs by Theorem 9.7.13.

We say that f is *ramified* at $x \in X$, if $\nu_{\mathcal{O}_{X,x}}(f^{\#,x}(t_y)) > 1$ and *unramified* if $\nu_{\mathcal{O}_{X,x}}(f^{\#,x}(t_y)) = 1$.

Example 9.8.4. Consider the affine example of [Example 9.8.2](#) for $a = 1$ and $b = 0$, that is, the morphism on spectra given by the ring homomorphism

$$k[x] \rightarrow k[x, y] / (y^2 - (x^3 - x))$$

Note that $x^3 - x = x(x - 1)(x + 1)$. There will be an exercise on the exercise sheet that says that being ramified is the same as the fiber being non-reduced (actually, this is not that hard to see from the definition). The fiber over $(x - c) \subseteq k[x]$ is

$$\operatorname{Spec} \left(k[x, y] / (y^2 - (c^3 - c)) \right)$$

This is non-reduced if and only if $c^3 - c = 0$, that is if $c = 0, 1$ or -1 . In these cases $y = 0$. So, the ramification points are:

$$(-1, 0), (0, 0), \text{ and } (1, 0).$$

9.9 GENERIC NOETHER NORMALIZATION

For the next section it is important to discuss how much of relative Noether normalization one can do that is, if R is a finitely generated A algebra over a domain, can one find a subring $S \subseteq R$ such that R is integral over S and that $S \cong A[x_1, \dots, x_n]$.

Note that in this situation, we have $\dim R = n + \dim A$ by [Theorem 7.4.1](#). Additionally, we also know that the fibers of R are finite over S .

Now, in the special case when R is a domain and $\dim R = \dim A$, then the only option is that $S = A$. However, then for the following example we see that Noether normalization of the above type does not exist:

Example 9.9.1. Consider $k[x, y] \hookrightarrow k\left[x, \frac{y}{x}\right]$. Applying Spec we obtain $f : X \rightarrow Y$, and the fiber over $(0, 0)$ has dimension 1.

However, if we allow ourselves a little localization in the base, then relative Noether normalization does exist:

Theorem 9.9.2. *If $S \hookrightarrow R$ is an embedding of domains such that R is a finitely generated S -algebra, then there is an element $f \in S$ such that R_f admits a Noether-normalization over S_f .*

Proof. Idea: You just do the classical Noether normalization and whenever you have to divide with something, then you put that into f .

We let you figure out the details. □

9.10 CHEVALLEY'S THEOREM

Remark 9.10.1. We start this section with a little hint to Exercise II.3.19 of Hartshorne. We suggest an alternative approach to the algebraic statement of Exercise II.3.19.b.

First, reduce by topological arguments and Noetherian induction to proving that:

$$\text{the image of a dominant map contains a non-empty open subset} \quad (9.10.a)$$

This is basically point (a) and (c) of the Hartshorne exercise.

We suggest that instead of the algebraic statement in Hartshorne you use for showing point (b) a relative Noether normalization (see [Theorem 9.9.2](#)). Use then that finite morphisms are surjective (for example by the going up theorem).

Remark 9.10.2. We also comment on how Chevalley's theorem states a property of varieties/schemes that does not happen even in complex differential geometry. In fact, if $f : X \rightarrow Y$ is a continuous and dominant map between topological manifolds with X compact, then $f(X)$ would be compact and hence closed, and then by the dominant assumption $f(X) = Y$ would need to hold. With other words, if X is assumed to be compact and f dominant, then Chevalley's theorem holds for any type of geometries by obvious reasons.

The corresponding notion to compactness in algebraic geometry is projectivity. The power of algebraic geometry lies in the fact that Chevalley's theorem does not need compactness, that is, a projectivity assumption on X .

We conclude this remark by showing that even if $f : X \rightarrow Y$ is a dominant holomorphic map of holomorphic manifolds, the Chevalley's theorem fails as soon as X is not assumed to be compact. Consider $\mathbb{C}$ and $\mathbb{Z}$ with the additive group structures, and $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$ with the multiplicative group structure. These are commutative groups, and we obtain the following homomorphism

$$\alpha : \mathbb{C} \rightarrow T := \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z} \cdot i \cong (\mathbb{R}/\mathbb{Z}) \oplus (\mathbb{R}/\mathbb{Z}) \cong S^1 \times S^1$$

$\uparrow$

$\mathbb{R}/\mathbb{Z} \cong S^1, \text{ because for the homomorphism } \phi : \mathbb{R} \rightarrow S^1 \text{ given by } x \mapsto e^{2\pi i x}, \text{ we have } \ker \phi = \mathbb{Z}$

The homomorphism α is a holomorphic covering map, that is, the preimages of small enough open sets are countably many biholomorphic disjoint copies of the original open sets. Note additionally that for $U := \{ a + bi \in \mathbb{C} \mid a, b \in [0, 1) \}$, the restriction $\alpha|_U$ is a holomorphic bijection (but not a homeomorphism, let alone a biholomorphism).

Fix now $d \in \mathbb{R} \setminus \mathbb{Q}$, and let $f : X := \mathbb{C} \rightarrow Y := T \times T$ be defined by $z \mapsto (\alpha(z), \alpha(\frac{1}{d} \cdot z))$. Fix then also $t \in T$, and choose $a, b \in \mathbb{R}$ such that $\alpha(\frac{a+ib}{d}) = t$. Then:

$$\left\{ z \in U \mid (\alpha(z), t) \in f(X) \right\} = \left\{ \underbrace{\{a + dn\}}_{\text{fractional part}} + i \underbrace{\{b + dm\}}_{\text{fractional part}} \mid n, m \in \mathbb{Z} \right\} \quad (9.10.b)$$

By basic number theory, as d is irrational, $\{ \{a + dm\} \mid m \in \mathbb{Z} \}$ is dense in $[0, 1)$. Therefore, by (9.10.b), $(T \times \{t\}) \cap f(X)$ is countable, but dense in $T \times \{t\}$. In particular it is dense, but it does not contain any open set. It follows that $f(X)$ is also dense in $T \times T$ but it does not contain any open set. Hence, f is an example of a holomorphic map between holomorphic manifolds which is dense but its image does not contain any open sets.

9.11 THE DIMENSION OF FIBERS

Again we suggest some changes to Exercise II.3.22 of Hartshorne. First, one can show the exercise for finite type morphisms in general. Second, instead of the hint of point (c), you can use a relative Noether normalization (see Theorem 9.9.2).

You should also use that generically finite dominant morphisms are finite over an open set (Exc. II.3.7 of Hartshorne). And you should also use Chevalley's theorem plenty of times (Exc. II.3.19).

9.11.1 Regularity of the geometric generic fiber vs. of general fibers

This is a hint to an exercise on the exercise sheet. Consider a finite type morphism $f : X \rightarrow Y$. That is, locally f fits into a commutative diagram as follows

$$\begin{array}{ccc} \text{Spec } A = X & \xrightarrow[\text{closed embedding}]{\iota} & \mathbb{A}_Y^n = \text{Spec } B[x_1, \dots, x_n] \\ & \searrow f & \downarrow \pi \\ & & Y = \text{Spec } B \end{array}$$

Let I be the kernel of $B[x_1, \dots, x_n] \twoheadrightarrow A$. If we assume that B is Noetherian, then $I = (f_1, \dots, f_r)$.

By Exc. II.3.22. we may also assume that all fibers of f have the same dimension, say d . Set $c = n - d$. Let $F_1, \dots, F_s$ be the determinants of all the $c \times c$ minors of the Jacobian matrix $\left(\frac{\partial f_j}{\partial x_i}\right)$. Consider then

$$Z = \operatorname{Spec} \left(B[x_1, \dots, x_n] / (f_1, \dots, f_r, F_1, \dots, F_s) \right).$$

Then, Z has a $\overline{\operatorname{a-Frac}(B)}$ rational point if and only if the geometric generic of f is not regular. And for some $m \in \operatorname{Spec} A$ maximal ideal with A/m algebraically closed, Z_m is not empty if and only if X_m is not regular.

Use now that Z has a $\overline{\operatorname{a-Frac}(B)}$ rational point if and only if $Z_\eta \neq \emptyset$, and then deduce that this happens if and only if there is a non-empty open set of Y over which the closed fibers Z_m are not-empty.

Use now that a constructible subset of a scheme of finite type over an algebraically closed field k is not-empty if and only if it has a k -rational point (this will be an exercise).

End of 15.
class, on
10.11.2020.

Chapter 10

Coherent sheaves

10.1 EXAMPLES

10.1.1 Sheaves of abelian groups that do not admit structures of sheaves of $\mathcal{O}_X$ -modules.

Example 10.1.1. Let $X = \operatorname{Spec} \mathbb{F}_p[x]$, and let $\mathcal{F} = (\mathbb{F}_2)_X$ be the constant sheaf on X , and assume that $p \neq 2$. We claim that $\mathcal{F}$ cannot be endowed with a structure of an $\mathcal{O}_X$ -module. Indeed, if it was possible, then $\mathcal{F}(X) = \mathbb{F}_2$ would be an $\mathcal{O}_X(X) = \mathbb{F}_p[x]$ -module. However by considering a composition series we see that all finite $\mathbb{F}_p[x]$ -modules have p^n elements for some integer $n > 0$.

Example 10.1.2. Let $X = \operatorname{Spec} \mathbb{F}_p[x]$, and let $\mathcal{F} = (\mathbb{F}_p)_X$ be the constant sheaf on X . We claim that $\mathcal{F}$ cannot be endowed with a structure of an $\mathcal{O}_X$ -module. Indeed, assume that $\mathcal{F}$ admits an $\mathcal{O}_X$ -module structure. Then $\mathcal{F}(X) = \mathbb{F}_p$ is an $\mathcal{O}_X(X) = \mathbb{F}_p[x]$ -module, that is, $\mathcal{F}(X) \cong \mathbb{F}_p[x] / (x - c)$ for some $c \in \mathbb{F}_p$. Let $U = X \setminus \{(x - c)\}$.

$$\begin{array}{ccccccc}
 0 & = & \rho_{X,U}^{\mathcal{F}}(0) & = & \rho_{X,U}^{\mathcal{F}}(1 \cdot (x - c)) & = & \rho_{X,U}^{\mathcal{F}}(1) \cdot \rho_{X,U}^{\mathcal{O}_X}(x - c) = \underbrace{1}_{\substack{\uparrow \\ \in \mathcal{F}(U) = \mathbb{F}_p}} \cdot \underbrace{x - c}_{\substack{\uparrow \\ x - c \in \mathbb{F}_p[x, (x - c)^{-1}] \text{ is invertible}}} \neq 0 \\
 \uparrow & & & & \uparrow & & \uparrow \\
 \boxed{\rho_{X,U}^{\mathcal{F}} \text{ is a homomorphism} \\ \text{of abelian groups}} & & \boxed{\mathcal{F} \text{ is a sheaf of } \mathcal{O}_X\text{-modules}} & & & & \boxed{\in \mathbb{F}_p[x, (x - c)^{-1}]}
 \end{array}$$

This is a contradiction.

10.1.2 Naive tensor product is not a sheaf

In this section we will learn about the following sheaves on $X = \mathbb{P}_k^1$: $\mathcal{O}_X(1)$ and $\mathcal{O}_X(-1)$, such that $\Gamma(X, \mathcal{O}_X(1)) \cong \mathbb{A}_k^2$, $\Gamma(X, \mathcal{O}_X(-1)) = 0$ and $\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-1) \cong \mathcal{O}_X$. This shows that the naive tensor product is not a sheaf, or equivalently the natural pre-sheaf homomorphism from the naive tensor product to the actual tensor product is not an isomorphism, because when we apply global sections it is also not an isomorphism:

$$\Gamma(X, \mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-1)) = \mathcal{O}_X(X) = k \neq 0 = k^{\oplus 2} \otimes_k 0 \cong (\mathcal{O}_X(1))(X) \otimes_{\mathcal{O}_X(X)} (\mathcal{O}_X(-1))(X).$$

10.1.3 Sheaves of $\mathcal{O}_X$ -modules that are not quasi-coherent

Let $\iota : U = D(x) = \operatorname{Spec} k[x, x^{-1}] \hookrightarrow \operatorname{Spec} k[x] = \mathbb{A}_k^1 = X$ be the open embedding, then $\mathcal{F} = \iota_! \mathcal{O}_U$ is a sheaf of $\mathcal{O}_X$ -modules, but it is not quasi-coherent. Indeed, for a coherent sheaf $\mathcal{G}$ on X , we have $\mathcal{G}(X) \neq 0$, however $\mathcal{F}(X) = 0$.

10.2 SHEAF HOMOMORPHISMS DEFINED ON BASES OF TOPOLOGIES

The following lemma is the key to make many of the proofs by handwavings in Section 5 precise.

Lemma 10.2.1. SHEAF HOMOMORPHISMS CAN BE DEFINED ON BASES OF TOPOLOGIES. *Let $\mathcal{B}$ a basis of open sets of a topological space X and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups (resp. sheaves of $\mathcal{O}_X$ -modules) on X . For every $U \in \mathcal{B}$, let $\phi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ be a homomorphism of abelian groups (resp. of $\mathcal{O}_X(U)$ -modules). Assume that for every $V, U \in \mathcal{B}$ such that $V \subseteq U$ the restriction morphisms commute with ϕ_U and ϕ_V , that is, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ \rho_{U,V}^{\mathcal{F}} \downarrow & & \downarrow \rho_{U,V}^{\mathcal{G}} \\ \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \end{array}$$

Show that then

- (1) There exists a unique homomorphism $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves of abelian groups (resp. of sheaves of $\mathcal{O}_X$ -modules), such that $\alpha_U = \phi_U$.
- (2) If all ϕ_U are injective, then so is α .
- (3) If all ϕ_U are surjective, then so is α .
- (4) In particular if all ϕ_U are isomorphisms, then so is α .

Proof. This will be an exercise on the exercise sheet. (Idea: if Z is a general open set, then cover it with $\{U \in \mathcal{C}\}$ for some subset $\mathcal{C} \subseteq \mathcal{B}$. Additionally, for each pair $U, V \in \mathcal{C}$, find a subset $\mathcal{D}_{U,V} \subseteq \mathcal{B}$ such that $\{W \in \mathcal{D}_{U,V}\}$ is an open cover of $U \cap V$. Then by the sheaf axioms we have

$$\mathcal{F}(Z) = \left\{ \prod_{U \in \mathcal{C}} s_U \in \prod_{u \in \mathcal{C}} \mathcal{F}(U) \mid \forall U, V \in \mathcal{C}, \forall W \in \mathcal{D}_{U,V} : s_U|_W = s_V|_W \right\}$$

and the same one for $\mathcal{G}(U)$.

Then it is obvious how to define α_Z . The extremely tedious part of the statements is that it is independent of the choice of $\mathcal{C}$. One way out of this is to choose $\mathcal{C}$ to be the set of all elements of $\mathcal{B}$ that is contained in Z , that is,

$$\mathcal{C} = \{ U \in \mathcal{B} \mid U \subseteq Z \}$$

Similarly, the only canonical choice of $\mathcal{D}_{U,V}$ is the following:

$$\mathcal{D}_{U,V} = \{ W \in \mathcal{B} \mid W \subseteq U \cap V \}$$

The above choices of $\mathcal{C}$ and $\mathcal{D}_{U,V}$ are canonical, and hence one does not have to worry about well-definedness.)

□

10.3 PROPOSITION II.5.2

Unfortunately, many details are swept under the rug in Proposition II.5.2 of Hartshorne. In fact, the proof presented in Hartshorne gives a horrible false impression that certain false argument are correct. For example that two sheaves are isomorphic if their stalks are isomorphic. This is one of the key mistakes one can make: in fact one needs to have a homomorphism of sheaves between the two sheaves. If that homomorphism is an isomorphism on stalks, then we can say that the two sheaves are isomorphic. However, there are many sheaves for which the stalks are isomorphic and still these sheaves are non-isomorphic as there is no homomorphism between them with the above properties.

10.3.1 Proposition II.5.2.c of Hartshorne

Here as the corresponding pre-sheaf on the right side is already a sheaf, one can use [Lemma 10.2.1](#) directly. One takes the basis given by open sets and notes that on $D(f)$ the sections of the sheaf on the left side is

$$\left(\bigoplus_i M_i \right) \otimes_A A_f$$

and on the right side it is

$$\bigoplus_i (M_i \otimes_A A_f)$$

These are naturally isomorphic, and the isomorphism is compatible with further localization. Hence, we can apply [Lemma 10.2.1](#).

10.3.2 Proposition II.5.2.b of Hartshorne

This is much trickier, as the sheaf on the right side is the sheafification of the naive pre-tensor product. Let us denote the latter by $\otimes^{\text{pre}}$. As in the previous point we can produce a natural isomorphism

$$(M \otimes_A N)_f = \left(\widetilde{M \otimes_A N} \right) (D(f)) \rightarrow \left(\tilde{M} \otimes_{\mathcal{O}_X}^{\text{pre}} \tilde{N} \right) (D(f)) = M_f \otimes_{A_f} N_f \quad (10.3.a)$$

This isomorphism sends $\frac{m \otimes n}{f^i}$ to $\frac{m}{f^i} \otimes n = m \otimes \frac{n}{f^i}$.

Compose (10.3.a) then with the sheafification, and we obtain a homomorphism

$$\left(\widetilde{M \otimes_A N} \right) (D(f)) \rightarrow \left(\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \right) (D(f))$$

which is again compatible with restrictions to open sets. This yields a global homomorphism $\alpha : \widetilde{M \otimes_A N} \rightarrow \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$ we want to show that this is an isomorphism. We cannot argue that it is an isomorphism on principal open sets as sheafification might have changed that.

However, *sheafification is an isomorphism on stalks*. So, it is enough to verify that α is an isomorphism on stalks. So, let $p \in \text{Spec } A$, and note that as sheafification is an isomorphism on stalks, α_p is the direct limit of the homomorphisms (10.3.a) for $p \in D(f)$. It is not hard to see that this direct limit is nothing else but the natural map $(M \otimes_A N)_p \rightarrow M_p \otimes_{A_p} N_p$ defined by the same formula as (10.3.a). Hence in particular it is an isomorphism.

(The above argument is still an outline, we let you fill in the details, for example to prove that $\varinjlim_{f \notin p} M_f \otimes_{A_f} N_f \cong M_p \otimes_{A_p} N_p$ as induced by the localization homomorphisms $N_f \rightarrow N_p$ and $M_f \rightarrow M_p$)

10.3.3 Proposition II.5.2.d of Hartshorne

This is again similarly to point (c) easier as there is no sheafification involved. One needs to use that localization by $f \in A$ as an A -module is naturally isomorphic to localization by f as a B -module.

10.3.4 Proposition II.5.2.e of Hartshorne

This is particularly messy to work out in full details, as it involves double sheafification (one when passing from $\tilde{M}$ to $\tilde{\tilde{M}}$, and one by passing from $f^{-1}\tilde{M}$ to $f^*\tilde{M}$). So, the idea of the solution is as in point (b) above. one constructs a homomorphism on principal open sets to the corresponding presheaves, and then to the actual sheaves. This then yields a global homomorphism. We prove that it is an isomorphism by using that sheafification does nothing on stalks, and that the constructed homomorphisms on principal open sets induce an isomorphism on stalks after taking direct limit.

We leave the details as homework for the interested reader (it will not be asked in the exam).

10.4 ON THE DEFINITION OF COHERENT SHEAF

While the definition of quasi-coherent sheaf given by Hartshorne is correct, the one for coherence given works fine only for locally Noetherian schemes.

Historically, however, the notion of coherence for sheaves actually originated in complex analytic geometry (principally in the work of Cartan and Oka) where the ring $\mathcal{O}_{\mathbb{C}^n}^{\text{hol}}(\mathbb{D})$ of holomorphic functions on a polydisk is not Noetherian, but Oka showed it still satisfy certain finiteness hypothesis (coherence indeed). The notion of coherence was later brought into algebraic geometry by J.P. Serre in his famous *Faisceaux algébriques cohérents* (FAC) article. This is somehow hinted in Exc II.5.4 of Hartshorne.

The following is the general definition of quasi-coherence for locally ringed spaces:

Definition 10.4.1. Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{F}$ be a $\mathcal{O}_X$ -module. We say that $\mathcal{F}$ is *quasi-coherent* if for every point $x \in X$ there exists a neighbourhood U of x such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a morphism of $\mathcal{O}_U$ -modules $\varphi: \mathcal{O}_U^{(I)} \rightarrow \mathcal{O}_U^{(J)}$.

Clearly $\mathcal{O}_X$ is a quasi-coherent module by definition.

It is important to remark that in the case of schemes, quasi-coherence detects if the $\mathcal{O}_X$ -module comes from a module over ring (so recovering the definition given by Hartshorne):

Proposition 10.4.2. Let A be a ring with $X = \text{Spec}(A)$ and let $\mathcal{F}$ be an $\mathcal{O}_X$ -module. Then the following are equivalent:

- (1) $\mathcal{F}$ is quasi-coherent;
- (2) for every $f \in A$, the canonical map $\mathcal{F}(X)_f \rightarrow \mathcal{F}(D(f))$ is an isomorphism of A_f -modules;
- (3) there exists a module M on A such that $\mathcal{F} \simeq \widetilde{M}$.

Proof. The only difficult part is (1) $\Rightarrow$ (2). We start by taking a collection of principal open sets $D(h_k)_{k \in K}$ such that $M|_{D(h_k)} \simeq \text{coker}(\varphi_k)$. It is possible to show that $M|_{D(h_k)} \simeq \widetilde{M}_k$ for $M_k = \mathcal{F}(D(h_k))$ as A_{h_k} -module (as cokernel of A_{h_k} -locally free modules). As X is quasi-compact we can suppose K is finite. We are now ready to prove (2).

We start with injectivity. Let $s \in \mathcal{F}(X)_f$ such that $s|_{D(f)} = 0$. For any $k \in K$ consider $s_k \in \mathcal{F}(D(h_k))$ such that $s|_{D(h_k)} = s_k$. As $s_k|_{D(fh_k)} = 0$, we conclude there exists $m_k > 0$ such that $f^{m_k} s_k = 0$. Take $m = \sup m_k$. Then clearly $f^m s = 0$ (as $f^m s|_{D(h_k)} = 0$), so $s = 0$.

We now show surjectivity. Let $s \in \mathcal{F}(D(f))$. Take $s|_{D(fh_k)}$ and note by construction of the covering that there exists $s_k \in \mathcal{F}(D(h_k))$ and $m_k > 0$ such that $f^{m_k} s|_{D(fh_k)} = s_k|_{D(fh_k)}$. Taking $M = \sup m_k$, we have $f^M s|_{D(fh_k)} = f^{M-m_k} s_k|_{D(fh_k)}$ (and clearly $f^{M-m_k} s_k$ is a global section on $D(h_k)$). This says that $f^{M-m_k} s_k$ coincides with $f^{M-m_j} s_j$ on $D(fh_k h_j)$. This implies that there exists $m(k, j)$ such that $f^{m(k, j)} (f^{M-m_k} s_k - f^{M-m_j} s_j) = 0$ on $D(h_k h_j)$. Take $N \gg 0$ such that $f^N (f^{M-m_k} s_k - f^{M-m_j} s_j) = 0$ on $D(h_k h_j)$ for all i, k . From this, we deduce there exists a section $t \in \mathcal{F}(X)$ such that $t|_{D(h_k)} = f^{N+M-m_k} s_k$. In particular, $t|_{D(fh_k)} = f^{M+N} s|_{D(fh_k)}$. This implies that the image of $\frac{t}{f^{M+N}} \in \mathcal{F}(X)_f$ via the canonical map is s , concluding surjectivity. $\square$

The following is the general notion of coherence for a sheaf.

Definition 10.4.3. Let X be a locally ringed space and let $\mathcal{F}$ be an $\mathcal{O}_X$ -module. We say that $\mathcal{F}$ is *coherent* if it is finitely generated and if, for every open subset U of X , and every finite family s_i of elements of $\mathcal{F}(U)$, the kernel of the associated morphism $\varphi: \mathcal{O}_X^I \rightarrow \mathcal{F}$ is finitely generated.

Note in particular that a coherent sheaf is locally finitely presented (but coherence is a stronger notion).

Example 10.4.4. There are at least two reasons why coherent sheaves are important. First, it is pretty clear that the previous definition is concerned on certain ‘boundedness’ of sections of sheaves, which will make their algebraically treatment (hopefully) simpler.

The second is that the category of coherent sheaves form an abelian category (don’t think we will have time to prove it using the previous definition) and thus we can use machinery from homological algebra on it. Moreover, if $\mathcal{O}_X$ is coherent as a module over itself (look at [Example 10.4.9](#) for an example which is not) then the category of coherent sheaves form the smallest abelian category (not time to prove it using the previous definition) containing locally free sheaves. The ‘naive’ idea of choosing sheaves finitely generated or finitely presented unfortunately do not form an abelian category outside the Noetherian world.

This is to explain why Hartshorne definition is not well-behaved in the non-Noetherian setting. The reason is simply because his definition is not the correct one.

We specialise to the case of schemes.

Lemma 10.4.5. *Let $X = \operatorname{Spec}(A)$ be an affine scheme and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$ -module. Then $\mathcal{F}$ is a coherent $\mathcal{O}_X$ -module if and only if $\mathcal{F}(X)$ is a coherent A -module.*

Proof. Omitted. □

The following explains that our definition of coherence coincides with Hartshorne’s in the case of locally Noetherian schemes.

Proposition 10.4.6. *A Noetherian ring is coherent as a module over itself. In particular, if X is a locally Noetherian scheme then the following are equivalent*

- (1) $\mathcal{F}$ is coherent;
- (2) $\mathcal{F}$ is a finitely generated quasi-coherent $\mathcal{O}_X$ -module;
- (3) for any affine open $\operatorname{Spec}(A) \subset X$ we have $\mathcal{F}|_U = \widetilde{M}$ with M a finite A -module.

Example 10.4.7. There are examples of coherent, not Noetherian rings. One is $A = k[\{x_i\}_{i \in \mathbb{N}}]$ for k field. In this case $\mathcal{O}_{\operatorname{Spec}(A)}$ is coherent (reference?), even though the scheme is not Noetherian.

Let me add a more detailed explanation to why a geometer should care about the definition of coherence we presented.

Remark 10.4.8. Consider the sheaf of holomorphic function $\mathcal{O}_{\mathbb{C}^n}^{\text{hol}}$. It’s not difficult in this case to see that the stalk $\mathcal{O}_{\mathbb{C}^n,0}^{\text{hol}}$ is Noetherian (consequence of Weierstrass preparation theorem). However, a polydisk $\mathbb{D}$ the sheaf $\mathcal{O}_{\mathbb{C}^n}^{\text{hol}}(\mathbb{D})$ is not Noetherian. A deep (and beautiful) theorem of Oka states that $\mathcal{O}_{\mathbb{C}^n}$ is coherent as a module over itself, which can be thought a finiteness result on holomorphic functions. This was an important achievement in for the complex analysis in several variables.

We give a non-coherent example.

Example 10.4.9. The sheaf of $\mathcal{C}^\infty$ -functions on a smooth real manifold is not coherent. This is somehow one of the reason why complex analytic geometry has more parallelism to algebraic geometry than differential geometry. Take the famous

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

Consider the kernel $\ker(\varphi)$ of the (not surjective!) multiplication map $\varphi: \mathcal{C}_0^\infty \rightarrow \mathcal{C}_0^\infty$ given by $\varphi(g) = fg$. It is easy to see that $\mathfrak{m} \ker(\varphi) = \ker(\varphi)$ and that $\ker(\varphi)$ is not zero. So if $\ker(\varphi)$ were finitely generated, we would contradict Nakayama’s lemma.

10.5 COMPLEMENTING EXAMPLES TO PROP 5.6 - COR 5.10 OF HARTSHORNE

10.5.1 The most frequently used exact sequence of coherent sheaves

$\mathcal{L}$ is an invertible sheaf (note the statement of Exc II.5.7.c), Z is a closed subscheme, and then we consider

$$0 \longrightarrow \mathcal{L} \otimes \mathcal{I}_Z \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}|_Z = \iota_* \iota^* \mathcal{L} \longrightarrow 0 ,$$

where $\iota : Z \hookrightarrow X$ is the inclusion. Note that this is indeed exact.

10.5.2 The exact sequence of Prop 5.6 is not exact typically at the right end

Example 10.5.1. Let Z be the reduced subscheme structure on $\{ [1, 0], [0, 1] \} \in \mathbb{P}_k^1$. Then if we take the above exact sequence for $\mathcal{L} = \mathcal{O}_{\mathbb{P}_k^1}$, after applying $\Gamma(\mathbb{P}_k^1, -)$, we do not get a surjective map at the right end, as the map is a k -linear map

$$k = \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}_k^1}) \rightarrow H^0(Z, \mathcal{O}_Z) = k^2.$$

10.5.3 Example that in Proposition 5.8.c $f_*\mathcal{F}$ might not be coherent (only quasi-coherent)

Example 10.5.2. Take the structure morphism $f : \mathbb{A}_k^1 \rightarrow \text{Spec } k$. Then $f_*\mathcal{O}_{\mathbb{A}_k^1} = \widehat{k[x]}$, which is infinite dimensional as a vector-space over k .

10.6 REMARKS FOR PROP 5.12 OF HARTSHORNE

Here the proof should be clarified as we did in Prop 5.2. The issues are the same: tensor product and pull-back involves sheafification(s). We leave it to the reader to work out the details not mentioned in class.

10.7 REMARKS FOR LEMMA 5.14 AND PROPOSITION 5.15 OF HARTSHORNE

These are just the Proj version of Lemma 5.3 and Prop 5.4 of Hartshorne. Indeed, the two lemma establish very similar extension and annihilation properties, and the the propositions show that these are exactly the properties needed to show that the natural homomorphisms are isomorphisms.

10.8 REMARKS FOR COR 5.16 OF HARTSHORNE

In this course we defined a scheme to be projective over A if it is $\text{Proj } S$ for some $\mathbb{N}$ -graded ring S with $S_0 = A$. Now, the book actually does it differently. It gives the definition in Section 4 (which we did not cover), as a closed subscheme of $\mathbb{P}_A^n$. Then, here in Cor 5.16 it shows that the two definitions are equivalent.

10.9 REMARKS ABOUT THM 5.19

Remark 10.9.1. The “well-known criterion for integral dependence” cited towards the end of the proof is [Proposition 9.7.1](#).

Remark 10.9.2. Thm 5.19 implies that if X is a projective variety over a (not-necessarily closed) field k , then $H^0(X, \mathcal{O}_X)$ is a finite k -vector-space. Here projective variety means that it is an integral projective scheme over k .

Hence, in this case, we have that $H^0(X, \mathcal{O}_X)$ is an Artinian ring. If this ring was non-reduced then we would have non-reduced elements also over some affine open set, which would contradict the integrality of X . Hence $H^0(X, \mathcal{O}_X)$ is a reduced Artinian ring. However, then it is a product of fields. Again using the irreducibility of X we obtain that $H^0(X, \mathcal{O}_X)$ is a finite field extension of k . If k is even algebraically closed then it even follows that $H^0(X, \mathcal{O}_X) = k$.

End of 16.
class, on
12.11.2020.

10.10 GEOMETRIC VECTOR BUNDLES AND LINE BUNDLES

Modification to exercise II.5.18: Recall the definition of a geometric vector bundle from exercise II.5.18. This exercise shows that geometric vector bundles and locally free sheaves of finite rank are equivalent. Unfortunately the constructions in Hartshorne between the two notions are not inverses of each others, see point c of the exercise. This can be cured easily by modifying the construction of point c. So, we suggest that instead of point (c) of the exercise show that the following gives a one to one correspondence

$$\begin{array}{ccc} \{\text{geometric vector bundles over } Y\} & \longleftrightarrow & \{\text{locally free sheaves of finite rank over } Y\} \\ \downarrow \Psi & & \downarrow \Psi \\ f : X \rightarrow Y & \longmapsto & \{\text{sheaf of sections of } X \text{ over } Y\} \\ \text{Spec } S(E^*) & \longleftarrow & \mathcal{E} \end{array}$$

10.10.1 Geometric meaning of $\mathcal{O}_{\mathbb{P}^n}(-1)$

The associated geometric line bundle to $\mathcal{O}_{\mathbb{P}^n}(-1)$ is

$$\{ (x, y) \in \mathbb{P}^n \times k^{n+1} \mid y \in x \}$$

where the map to $\mathbb{P}^n$ is given by projection on the first coordinate. To see let $S = k[x_0, \dots, x_n]$ as a graded ring with the standard grading, and this consider the following exact sequence of

$$S^{\oplus(n+1)} \xrightarrow{(s_0, \dots, s_n) \mapsto \sum_{i=0}^n s_i x_i} S(1)$$

Applying $\text{Spec } S(\text{---})$ yields the above description of $\mathcal{O}_{\mathbb{P}^n}(-1)$.

10.10.2 The difference between $\mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{O}_{\mathbb{P}^1}(1)$

Let $S = k[x, y]$ with the standard grading. Then $\mathbb{P}^1 = \text{Proj } S$. We have $\mathcal{O}_{\mathbb{P}^1}(-1) = \widetilde{S(-1)}$, and $\mathcal{O}_{\mathbb{P}^1}(1) = \widetilde{S(1)}$. Let $U = D(y)$ and $V = D(x)$. Then,

$$\mathcal{O}_{\mathbb{P}^1}(-1)|_U = \widetilde{S(-1)}_{(y)} = k \left[\frac{x}{y} \right] \frac{1}{y}$$

and

$$\mathcal{O}_{\mathbb{P}^1}(-1)|_V = \widetilde{S(-1)}_{(x)} = k \left[\frac{y}{x} \right] \frac{1}{x}$$

Additionally, the gluing morphisms are given by the embeddings

$$k \left[\frac{x}{y} \right] \frac{1}{y} \hookrightarrow k \left[\frac{x}{y}, \frac{y}{x} \right] \frac{1}{y} = k \left[\frac{x}{y}, \frac{y}{x} \right] \frac{1}{x} \hookleftarrow k \left[\frac{y}{x} \right] \frac{1}{x}$$

(Note that here we are working in the degree 1 part of the localization of S at all the homogeneous elements, as all our section rings and local rings naturally live inside it, by construction and by Prop 5.11 of Hartshorne).

Hence, global sections on U are sums of terms of the form $\frac{x^i}{y^{i+1}}$ and over V they are sums of terms of the form $\frac{y^i}{x^i}$, for $i \geq 0$. So, the only ones that glue is 0, and hence $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$, as we know from Prop 5.13 of Hartshorne.

On the other hand, if we replace -1 by 1 , then we get

$$k \left[\frac{x}{y} \right] y \hookrightarrow k \left[\frac{x}{y}, \frac{y}{x} \right] y = k \left[\frac{x}{y}, \frac{y}{x} \right] x \hookleftarrow k \left[\frac{y}{x} \right] x$$

Note that $\frac{x}{y}y = x$ and that $\frac{y}{x}x = y$. Hence, the global section $\frac{x}{y}y$ on U and the global section x on V glue to give a global section on $\mathbb{P}^1$. Same happens with y and $\frac{y}{x}x$. Hence we have two k -linearly independent global section in $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, and indeed this is a vector space of dimension 2.

Chapter 11

Divisors

This is about Section II.6 of Hartshorne. We skip in this section some proofs, check with the lecture notes/recordings about what was skipped, and hence not asked in the exam.

Chapter 12

Morphisms to $\mathbb{P}^n$ and linear systems

This is about Section II.7 of Hartshorne. We cover this section only partially, only up to linear systems. We also skip even in this part some proofs. Check with the lecture notes/recordings about what was skipped, and hence not asked in the exam.

Chapter 13

Differentials

This is about Section II.8 of Hartshorne. We cover this section only briefly. We note/cover the following from Hartshorne

- (1) the definition of $\Omega_{B/A}$
- (2) If X is a scheme over Y , then there is $\Omega_{X/Y}$, which is...
- (3) the regularity criterion
- (4) $\omega_{X/k} = \wedge^{\dim X} \Omega_{X/k}$.

Remark 13.0.1. Positivity (ampleness) of $\omega_{X/k}$ corresponds to negative curvature with differential geometric terminology. Similarly negativity of $\omega_{X/k}$, that is positivity of $\omega_{X/k}^{-1}$ corresponds to positive curvature.

One of the main goal of algebraic geometry is to classify varieties, and the above types are the fundamental building blocks. See <https://www.epfl.ch/labs/cag/classification-theory-of-algebraic>

Chapter 14

Cohomology

Chapter 15

Cohomology of a Noetherian scheme

Chapter 16

Čech cohomology

Chapter 17

Riemann-Roch on curves

Chapter 18

Commutative algebra statements not covered in the course

- (1) any localization of a regular Noetherian local ring is a regular Noetherian local ring, and
- (2) a regular local ring is a UFD, and hence it is integrally closed.
- (3) if R is a finitely generated k -algebra which is a domain, then so is its integral closure in L , where L is a finite field extension of $\text{Frac}(R)$.
- (4) For a given singularity condition $\mathcal{P}$:
 - (i) $\mathcal{P}$ holds for a ring R if and only if R_p holds for all $p \in \text{Spec } R$, (this is a different theorem for all $\mathcal{P}$, and for most of them we did not show it)
 - (ii) $f : X \rightarrow Y$ is a morphism of finite type with Y integral, then there is a dense open set $U \subseteq Y$ such that either for every geometric point $\text{Spec } L \rightarrow U$ the property $\mathcal{P}$ holds for X_L or for every geometric point $\text{Spec } L \rightarrow U$ the property $\mathcal{P}$ does not for X_L (this is a different theorem for all $\mathcal{P}$, and for most of them we did not show it, we did $\mathcal{P} = \text{regular}$).