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AG II - Schemes and sheaves

Exercises – week 9

Exercise 1. *Nullstellensatz* via *Chevalley.* Let k be a field and $\mathfrak{m}$ be a maximal ideal of $k[x_1, \ldots, x_n]$.

(1) Show by contradiction and using week 8, exercise 8 (which was a direct consequence of Chevalley), that $\mathfrak{p}_i = k[x_i] \cap \mathfrak{m}$ is maximal (so $\neq 0$) for each $i = 1, \ldots, n$.
Hint: If $\mathfrak{p}_i = 0$*, we have a dominant map* $\operatorname{Spec}(k(\mathfrak{m})) \to \mathbb{A}^1_k$.

The above is called *Zariski's lemma* and is the key to Nullstellensatz. Deduce from the lemma proved in item (1) the following direct consequences.

(2) Deduce that
$$\mathfrak{m} = (\mathfrak{p}_1, \ldots, \mathfrak{p}_n)$$
and that $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a finite field extension of k.

(3) Let $A \to B$ a k-algebra map between finite type k-algebras. Show that $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ carries closed points to closed points.

(4) Deduce that any finite type finite type k-algebra A is *Jacobson*, meaning that the nilradical (intersection of all primes, see week 3 exercise 1) of A is equal to the intersection of maximal ideals of A.
Hint: for f *not nilpotent, use the preceding point with* $A \to A_f$.

Exercise 2. *Dual.* Let $\mathcal{E}$ be locally free sheaf of finite rank[1] on a ringed space $(X, \mathcal{O}_X)$ and $\mathcal{F}$ an $\mathcal{O}_X$-module. We define $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$.

(1) Show that $\mathcal{E}^\vee$ is a locally free sheaf of finite rank, the same $\mathcal{E}$.
(2) Show that there is a natural isomorphism $\mathcal{E} \to \mathcal{E}^{\vee\vee}$.
(3) Show that there is a natural isomorphism $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$.

Exercise 3. *Compatibilities between* f^*, f_* *and* $\otimes$. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.

(1) Let $\mathcal{G}$ and $\mathcal{H}$ be sheaves of $\mathcal{O}_Y$-modules. Show that there is a natural isomorphism
$$f^*(\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H}) \cong f^*(\mathcal{G}) \otimes_{\mathcal{O}_Y} f^*(\mathcal{H}).$$

(2) (*Projection formula.*) Let $\mathcal{F}$ be an $\mathcal{O}_X$ module and $\mathcal{E}$ be a finite locally free sheaf on $\mathcal{O}_Y$. Show that there is a natural isomorphism
$$\mathcal{E} \otimes_{\mathcal{O}_Y} f_*\mathcal{F} \to f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

[1] For $n \in \mathbb{N}$ a locally free sheaf of rank n is an $\mathcal{O}_X$-module which is locally isomorphic to $\mathcal{O}_U^{\oplus n}$ where U ranges in an open cover of X.

Exercise 4. *Fibre dimension (of coherent sheaves).*
Let X be a Noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. Let

$$\varphi\colon X \to \mathbb{N}$$

be defined as $\varphi(x) = \dim_{k(x)}(\mathcal{F} \otimes_{\mathcal{O}_X} k(x))$.
Nakayama's lemma may be useful for the following.

(1) Show that φ is upper semi-continuous meaning that for any $n \geq 0$

$$\{x \in X \mid \varphi(x) \geq n\}$$

is closed.
(2) If $\mathcal{F}$ is locally free and X connected show that φ is constant.
(3) Show that if X is reduced and connected show that $\mathcal{F}$ is locally free if and only φ is constant.

Exercise 5. *Fibre dimension (of finite type morphisms)* We recall some results along the way that you can assume.

Lemma 1 (Krull's height theorem)**.** *Let R be a Noetherian ring. Suppose that $\mathfrak{p}$ is a minimal prime of $(f_1, \ldots, f_n)$. Then*

$$\mathrm{ht}(\mathfrak{p}) \leq n.$$

(1) Let R be a Noetherian ring and $\mathfrak{p}$ be a prime ideal. Using Krull's height theorem, show by induction on the height that for every prime $\mathfrak{p}$ of height n there is $(f_1, \ldots, f_n) \subset \mathfrak{p}$ such that $\mathfrak{p}$ is a minimal prime of $(f_1, \ldots, f_n)$ and every minimal prime of $(f_1, \ldots, f_n)$ has height n.
(2) Let $f\colon X \to Y$ be a morphism between locally Noetherian schemes and $Y' \subset Y$ a closed irreducible subset. Show that for every irreducible component $Z \subset f^{-1}(Y')$ that dominates Y' we have

$$\mathrm{codim}(Z, X) \leq \mathrm{codim}(Y', Y).$$

Hint: This is a local problem so you can reduce to affines and use item (1).

Lemma 2. *Let k be a field, A be a finite type k-algebra which is also a domain and $\mathfrak{p} \in \mathrm{Spec}(A)$. Then*

$$\dim(A) = \mathrm{trdeg}_k(\mathrm{Frac}(A))$$
$$\text{and } \mathrm{codim}(\mathrm{Spec}(A/\mathfrak{p}), \mathrm{Spec}(A)) = \mathrm{ht}(\mathfrak{p}) = \dim(A) - \dim(A/\mathfrak{p}).$$

(3) Let $f\colon X \to Y$ be a map between finite type integral k-schemes. Show that for every $y \in f(X)$ and Z irreducible component of X_y we have

$$\dim(X) - \dim(Y) \leq \dim(Z) \leq \dim(X).$$

Hint: Use item (2) with $Y' = \overline{\{y\}}$. Use lemma 2 and the additivity of transcendence degree with $k \mid k(y) \mid K(Z)$. Namely

$$\mathrm{trdeg}_k(K(Z)) = \mathrm{trdeg}_k(k(y)) + \mathrm{trdeg}_{k(y)}(K(Z)).$$

(4) Let $f\colon X \to Y$ be a dominant map between finite type integral k-schemes. Show that there is an open dense $U \subset X$ such that for all $y \in f(U)$ we have $\dim(X_y) = \dim(X) - \dim(Y)$ and $f(U)$ is open.
Hint: show that you can reduce to the affine case $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *with* $t_1, \ldots, t_e \in B$*, where* $e = \dim(X) - \dim(Y)$*, such that* $t_1, \ldots, t_e$ *form a transcendence basis of* $K(X)$ *over* $K(Y)$*. Then factor the morphism by* $\operatorname{Spec}(A[t_1, \ldots, t_n])$*. Note that* $X \to \operatorname{Spec}(A[t_1, \ldots, t_e])$ *induces a finite morphism at fraction fields and that* $\operatorname{Spec}(A[t_1, \ldots, t_e]) \to \operatorname{Spec}(A)$ *is isomorphic to* $\mathbb{A}^e_A \to \operatorname{Spec}(A)$ *which is open by exercise 2. Use exercise 1.(2) to conclude.*
Remark. You are free to prove the following weaker version of the statement: show that there is an open dense $U \subset X$ such that for all $y \in f(U)$ we have $\dim(U_y) = \dim(X) - \dim(Y)$ and $f(U)$ is open.

(5) Let $f\colon X \to Y$ be a dominant map between finite type integral k-schemes. For $h \in \mathbb{N}$, let E_h be the set of points x of X such that there is an irreducible component of $X_{f(x)}$ with dimension at least h, which contains x. Show that E_h is closed.[2]
Hints: If $h \leq e$*, then* $E_h = X$ *by (3). If* $h > e$*, note that* $E_h \subset X \setminus U$ *where* U *is the open of item (4). Proceed by induction on the dimension of* X*.*

(6) Let $f\colon X \to Y$ be a closed map between finite type integral k-schemes. For $h \in \mathbb{N}$, let F_h be the set of points of y of Y such that there is an irreducible component of X_y with dimension at least h. Show that F_h is closed.
Hint: Show that $F_h = f(E_h)$*.*

Exercise 6. *Criterion of irreducibility.* This exercise uses results of the last exercise, see the hint for more details. Let k be a field.

(1) Suppose that $f\colon X \to Y$ is a map between finite type k-schemes, with Y being irreducible, such that every fiber is irreducible of a fixed dimension $d \geq 0$ (in particular f is surjective). Show that X is irreducible if
  (a) f is closed or,
  (b) X is equidimensional.
Hint: write $X = \bigcup_i X_i$ *the decomposition into irreducible components of* X*. There is at least one irreducible component, say* X_1*, such that* $f(X_1)$ *is dense. Write* $U_1 = X_1 \setminus \bigcup_{i \neq 1} X_i$*. Using irreducibility of fibers, show that for every* $y \in f(U_1)$ *we have* $X_y = X_{1,y}$*. Use exercise 2 and hand-in item (3) to get an open set* $V \subset f(U_1)$ *of* Y *such that every fiber at* $y \in V$ *has dimension* $\dim(X) - \dim(Y)$*. Show also that* $X_i \setminus X_1 \subset f^{-1}(Y \setminus V)$*. Deduce that* X_1 *is the only irreducible component with a dense image. Conclude if you suppose (b). If you suppose (a), show that* $f(X_1) = Y$ *and conclude.*

[2] The statement is true for any $f\colon X \to Y$ between X and Y finite type k-schemes without the dominant hypothesis. This can be shown by an easy reduction to the case of the exercise.

(2) Deduce that if X is an irreducible finite type k-scheme

$$X \times_k \mathbb{P}^n_k$$

is irreducible.