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AG II - Schemes and sheaves

Exercises – week 8

Exercise 1. *Closed subschemes.*

(1) Let X be a scheme. Let $\mathcal{I}$ be a quasi-coherent ideal sheaf. Let $Z = \operatorname{supp}(\mathcal{O}_X/\mathcal{I})$. Denote by $\iota : Z \to X$ the inclusion. Show that $(Z, \iota^*\mathcal{O}_X/\mathcal{I})$ is a scheme. In what follows, $V(\mathcal{I})$ denotes the above associated scheme.
Hint: This is a local question. So using that $\mathcal{I}$ is quasi-coherent you can reduce to the case where X is affine and $\mathcal{I}$ correspond to an ideal.

(2) Show that $V(\mathcal{I})$ is a closed subscheme of X.

(3) Show that

$$\{\text{Quasi-coherent ideals of } \mathcal{O}_X\} \longleftrightarrow \{\text{Closed subschemes of } X\}$$

sending $\mathcal{I}$ to $V(\mathcal{I})$ is a one-to-one correspondence.

(4) Let $\operatorname{Spec}(A)$ be an affine scheme. Show that

$$\{\text{Ideals of } A\} \longleftrightarrow \{\text{Closed subschemes of } \operatorname{Spec}(A)\}$$

sending $I \mapsto (\operatorname{Spec}(A/I) \to \operatorname{Spec}(A))$ is a one-to-one correspondence.
Hint: You may use the equivalence of categories between quasi-coherent sheaves of $\mathcal{O}_{\operatorname{Spec}(A)}$-modules and A-modules. For a proof which does not use this fact, see the solution of exercise 5, week 6.

Exercise 2. *Intersection of affine schemes.* Let X be a scheme and $U, V \subset X$ be open affine sub-schemes.

(1) Show that if X is separated then $U \cap V$ is affine.
Hint: Show that $U \cap V \cong X \times_{X \times X} (U \times V)$.

(2) Show that $U \cap V$ is not necessarily affine if X is not separated.
Hint: remember this open of an affine which is not affine? Play with this.

Exercise 3. *A map from a proper scheme to a separated scheme is closed.* Let $f\colon X \to Y$ be a map of S-schemes. Suppose that $Y \to S$ is separated.

(1) Show that the graph $(\mathrm{id}, f) = \Gamma_f\colon X \to X \times_S Y$ is a closed immersion.

(2) Let $Z \subset X$ a closed subscheme proper over S. Show that $f_{|Z}$ is closed.

Remark. This fact is analogue to the topological result that a continuous map from a compact topological space to a Hausdorff space is always closed.

Exercise 4. *Morphisms into separated schemes.* Let S be a scheme. Let $X \to S$ and $Y \to S$ be S-schemes. Suppose that X is reduced and $Y \to S$ separated. Show that two morphisms of S-schemes

$$f_1, f_2 \colon X \to Y$$

that coincide an open dense subset of X are equal.
Give counter-examples if one of the hypotheses is dropped.

Remark. This fact is analogue to the topological result that if two continuous morphisms to a Hausdorff space agree on a open dense then they actually agree everywhere.

Exercise 5. *Generically finite morphisms.*

(1) Let k be a field. If $k \to A$ is finite, show that every prime of A is maximal.

Let $f \colon X \to Y$ be a dominant morphism between integral schemes.

(2) If f is finite, show that $\dim(X) = \dim(Y)$.
Hint: reduce to the affine case. Then use going up and that the map is surjective. Use point (1) to deduce that if $A \to B$ is finite and the preimage of two primes in B is the same in A, then the two primes are not included in one another.
(3) If f is finite type and $K(Y) \subset K(X)$ is a finite extension of fields, show that there exists an non-empty open $U \subset Y$ such that $f \colon f^{-1}(U) \to U$ is a finite morphism.
Hint: first prove the case where both X and Y are affine and then battle to use this case to conclude.

Exercises 6, 7, and 8 are purely about the underlying topology of the schemes in question.

Exercise 6. *Projection from affine spaces.* Let R be a ring.

(1) Show that

$$\pi \colon \operatorname{Spec}(R[t]) \to \operatorname{Spec}(R)$$

is open. More precisely, if $f = \sum a_i t^i$ show that

$$\pi\left(D(f(t))\right) = \bigcup_i D(a_i).$$

(2) Let $g(t) \in R[t]$ be a monic polynomial and $f(t) \in R[t]$. Remark that $R[t]/g(t)$ is a free R-module of rank $\deg(g)$. Let $\chi(X) = \sum_i^n r_i X^i$ be the characteristic polynomial of the multiplication by $f(t)$ on $R[t]/g(t)$. Show that

$$\pi(D(f) \cap V(g)) = \bigcup_i^{n-1} D(r_i).$$

Exercise 7. *Chevalley's theorem.* Let X be a Noetherian topological space. A subset $T \subset X$ is called *constructible* if it can be written as a finite union of sets of the form $U \cap V^c$ where U and V are open sets.

(1) Show that if $X = \operatorname{Spec}(R)$ for a Noetherian ring R, a subset is constructible if and only if it can be written as a finite union of subsets of the form $D(f) \cap V(g_1, \ldots, g_m)$ with $f, g_1, \ldots, g_m \in R$.
(2) Show using exercise 6 that

$$\pi\colon \operatorname{Spec}(R[t]) \to \operatorname{Spec}(R)$$

sends constructible subsets to construtible subsets.
Hint: Show by induction on $\sum_i \deg(g_i)$ that if $f, g_1, \ldots, g_m \in R[t]$ are polynomials, the image of $D(f) \cap V(g_1, \ldots, g_m)$ is constructible. To conduct the induction step, consider α the leading coefficient of g_1. Break down the study on the open and closed $D(\alpha)$ and $V(\alpha)$ to reduce the sum of the degrees.
(3) Deduce *Chevalley's theorem.* Let $f\colon X \to Y$ be a finite type morphism between Noetherian schemes. Then f sends constructible subsets to constructible subsets.[1]

Remark. In general the topological image of a morphism of schemes can fail to be open or closed but in cases where Chevalley's theorem applies, it tells that it still not too far from it and manageable. In particular one can endow the image with a scheme structure.

Exercise 8. *An application of Chevalley's theorem.* Let $f\colon X \to Y$ be a finite type dominant map between Noetherian schemes with Y irreducible. Use Chevalley's theorem to show that the topological image $f(X)$ contains an open set.

Exercise to hand in. *Ramifications of some self maps of $\mathbb{P}^1$.* (Due 17 November, 18:00) Please write your solution in TeX.

- We say that a map of schemes $f\colon X \to Y$ is *finite locally free* if there is a covering of Y by open affines $\operatorname{Spec}(A_i)$, with preimage $\operatorname{Spec}(B_i)$, such that induced map $A_i \to B_i$ turns B_i into a finite free A_i-module. When for every i the dimension of B_i is the same, say d, we say that the map is *finite locally free of degree d.*
- We say that a finite locally free map $X \to Y$ is *ramified at $y \in Y$* if the geometric fiber $X_{\overline{y}}$ is not reduced.

(1) Show that the self map c_n from week 7, exercise 1 is finite locally free of degree n and identify it's ramification points.
(2) Let R be a ring. Show that the map induced on $\mathbb{P}^1_R = \operatorname{Proj}(R[x, y])$ by the R-algebra self map $x \mapsto ax + by$ and $y \mapsto cx + dy$ is an automorphism if $ad - bc \in R^\times$. We denote this map $m_{(a,b,c,d)}$.

[1] The generalization to non-Noetherian settings requires more careful definitions, but once these definitions are addressed the proof is the same.

If $R = \mathbb{C}$ and if we identify $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C}) = \mathbb{C} \cup \infty$, how is this map expressed on $\mathbb{C}$-rational points?

(3) Consider the composition

$$\mathbb{P}^1_{\mathbb{C}} \xrightarrow{c_n} \mathbb{P}^1_{\mathbb{C}} \xrightarrow{m_{(1,-1,1,1)}} \mathbb{P}^1_{\mathbb{C}} \xrightarrow{c_2} \mathbb{P}^1_{\mathbb{C}}.$$

Show it's finite locally free of fixed degree. What is the degree? What are the ramification points? Compute scheme theoretic fibers at all ramification points.