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AG II - Schemes and sheaves

Exercises – week 6

Exercise 1. *Normal schemes and normalization* An integral scheme X is said to be *normal* if every stalk $\mathcal{O}_{X,x}$ is integrally closed.

(1) Show that an affine integral scheme $\operatorname{Spec}(A)$ is normal if and only if A is normal ring.
(2) Show that an integral scheme is normal if and only for every closed point $x \in U$ the stalk $\mathcal{O}_{X,x}$ is integrally closed for every open affine $U \subset X$.[1]

The *normalization* of an integral scheme X is a scheme $\widetilde{X}$ together with a dominant map[2] $\nu\colon \widetilde{X} \to X$ such that for every dominant morphism from an integral normal scheme $f\colon Z \to X$ there exists a unique morphism $\overline{f}\colon Z \to \widetilde{X}$ with $\nu\overline{f} = f$. Therefore the normalization is unique up to unique isomorphism.

(3) Let A be an integral domain. Show that if $X = \operatorname{Spec}(A)$, then $\operatorname{Spec}(\widetilde{A})$ is the normalization of X if $A \to \widetilde{A}$ is the normalization of A.
(4) Show that every integral scheme admits a normalization.

Exercise 2. *Blow-ups.* Let k be an algebraically closed field. You can use the following.
Let $A = k[x_1, \ldots, x_n]/(f)$. Denote by $\partial_i f$ the derivative of f with respect to x_i. Then

$$\operatorname{Spec}(A) \text{ is regular } \iff V(f, \partial_1 f, \cdots, \partial_n f) = \emptyset.$$

Moreover $V(f, \partial_1 f, \cdots, \partial_n f)$ consists exactly of the non-regular points of $\operatorname{Spec}(A)$.

(0) Let R be a ring. Show that if $I = (f_0, \ldots, f_n)$ is generated by a regular sequence then $\mathrm{Bl}_I = V_+(X_i f_j - X_j f_i)$ in $\mathbb{P}^n_R = \mathbb{P}^n_{\mathbb{Z}} \times \operatorname{Spec}(R)$. (Use the lemmas in the blow-ups document from moodle)
(1) Show that blow-up of (x^2, y^2) in $\operatorname{Spec}(k[x, y])$ is not normal and that the blow-up of (x, y) is its normalization.
(2) Show that blow-up of (x^2, y) in $\operatorname{Spec}(k[x, y])$ is not regular. What are the regular points?[3]
(3) Show that $X = \operatorname{Spec}(k[x, y, z, w]/(xy - zw))$ is not regular. What are the regular points?

[1] For finite type k-schemes, this the same as saying every closed point of X. See week 10, exercise 1.
[2] A map is called *dominant* if the topological image of the map is dense.
[3] This investigation can be used to show that this blow-up is normal.

(4) Show that blow-ups of X at (x, y, z, w) and (x, z) are regular. We denote these blow-ups by $X_1 \to X$ and $X_2 \to X$.
Remark. This is another example where blow-ups resolves (=removes) singularities, as in 4.(3) of week 5.
(5) Compute fibers of (x, y, z, w) of $X_1 \to X$ and $X_2 \to X$.

Exercise 3. *Integrality/reducedness of* Proj. Let B be an $\mathbb{N}$-graded integral/reduced ring. Show that $\mathrm{Proj}(B)$ is an integral/reduced scheme.

Exercise 4. *Fibers.*

(1) Compute the fibers of the morphism
$$\mathrm{Spec}(\mathbb{Z}[x, y, z]/(2zx + 9y^2)) \to \mathrm{Spec}(\mathbb{Z}).$$
Which fiber is reduced ? Which fiber is integral ?
(2) Compute the fibers of the morphism, where p is a prime number
$$\mathrm{Spec}(\mathbb{Z}[x, y]/(xy^2 + p)) \to \mathrm{Spec}(\mathbb{Z}).$$
Which fiber is reduced ? Which fiber is integral ?

Exercise 5. *Properties under base change.* Let $f: X \to Y$ be a morphism of schemes. Which of the following properties are stable under base change? Prove the statement or provide a counter-example.

(1) f is an open immersion.
(2) f is a closed immersion.
(3) f is injective.
(4) f has integral fibers.
(5) f has reduced fibers.

Exercise 6. *An open of an affine is not neccesarly affine.* Let R be a non-zero ring. Show that $U = \mathrm{Spec}(R[x, y]) \setminus V(x, y)$ is not affine.
Hint: compute $\mathcal{O}(U)$ using an appropriate cover and the sheaf property.

Exercise to hand in. *Twistor* $\mathbb{P}^1$. (Due Sunday November 3, 18:00) Please write your solution in TEX.
Consider the following graded $\mathbb{R}$-algebra map $\sigma: \mathbb{C}[x, y] \to \mathbb{C}[x, y]$
$$f(x, y) \mapsto \overline{f}(-y, x),$$
where $\overline{(-)}$ means applying the complex conjugation to the coefficients of the polynomial. Denote by $\tau = \mathrm{Proj}(\sigma)$ the induced $\mathbb{R}$-scheme morphism of $\mathbb{P}^1_{\mathbb{C}}$.

(1) Recall that we can identify the $\mathbb{C}$-rational points $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C})$ as equivalences classes $[\alpha, \beta]$ of elements of $\mathbb{C}^2 \setminus 0$. We can also identify those with $\mathbb{C} \cup \infty$ with $[1, 0] \mapsto \infty$ and $[z, 1] \mapsto z$.
Describe τ on $\mathbb{C}$-rational points
$$\tau: \mathbb{P}^1_{\mathbb{C}}(\mathbb{C}) \to \mathbb{P}^1_{\mathbb{C}}(\mathbb{C})$$
with these two identifications of $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C})$.

(2) Show that $\tau^2 = \mathrm{id}$ schematically.

We now consider

$$\mathbb{C}[x,y]^\sigma = \{f(x,y) \in \mathbb{C}[x,y] \mid f(x,y) = \sigma(f(x,y)) = \overline{f}(-y,x))\}$$

the graded $\mathbb{R}$-algebra of σ-fixed points. We define the $\mathbb{R}$-scheme

$$\mathbb{P}^1_{\mathrm{tw}} = \mathrm{Proj}(\mathbb{C}[x,y]^\sigma)$$

and call it the *twistor* $\mathbb{P}^1$.

(3) Show that

$$\mathbb{C}[x,y]^\sigma = \mathbb{R}[ixy, x^2+y^2, i(x^2-y^2)].$$

(4) Show that $\mathbb{P}^1_{\mathrm{tw}} \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) \cong \mathbb{P}^1_{\mathbb{C}}$.

(5) Identify the kernel of the 2-homogeneous map of $\mathbb{R}$-algebras

$$\mathbb{R}[t_1,t_2,t_3] \to \mathbb{C}[x,y]^\sigma$$

sending $t_1 \mapsto 2ixy$, $t_2 \mapsto x^2+y^2$, $t_3 \mapsto i(x^2-y^2)$.

(6) Show that there is no $\mathbb{R}$-rational points of the twistor $\mathbb{P}^1$, *i.e.*

$$\mathbb{P}^1_{\mathrm{tw}}(\mathbb{R}) = \emptyset.$$

Remark. Some explanations and motivation.

(1) The map $\tau\colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ from the first part of the exercise is what we call a *descent data*. Like saying that a sheaf can be described as a collection of sheaves on opens sets which forms a covering with isomorphisms on intersections satisfying the cocycle condition, here we specify an $\mathbb{R}$ scheme as a $\mathbb{C}$-scheme with some special involution. Here the *cover* taken is the Galois cover $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ and the involution property has to do with the fact that the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is of order 2.

(2) The *descended* $\mathbb{R}$-scheme that we get is a $\mathbb{R}$-conic (curve described by a degree 2 equation in $\mathbb{P}^2_{\mathbb{R}}$) which has the strange property that it has no $\mathbb{R}$-points. In general if k is a field and $C \subset \mathbb{P}^2_k$ is a regular conic with admits at least one k-rational point (which here is not the case for $\mathbb{P}^1_{\mathrm{tw}}$), then it is actually isomorphic to $\mathbb{P}^1_k$. This can be understood *via* the nice method of stereographic projection. See here for for a nice explanation of this method.

(3) The $\mathbb{R}$-scheme $\mathbb{P}^1_{\mathrm{tw}}$ appears in cohomology theory. Namely one can show that vector bundles on this scheme are intimately related to *Hodge structures*. Singular cohomology groups of projective varieties admits Hodge structures, that's why we care about this kind of linear algebraic objects.