Dr. Stefano Filipazzi
Dr. Alapan Mukhopadhyay
Léo Navarro Chafloque

EPFL, fall semester 2024
AG II - Schemes and sheaves

Exercises – week 14

Exercise 1. *A short exact sequence.* This first exercise is just a recall. If you don't understand why this is true, understand it.
Let $\iota\colon D \to X$ be an effective Cartier divisor on a integral scheme X. As there is a short exact of sheaves

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to \iota_*\mathcal{O}_D \to 0.$$

In consequence there is a long exact sequence in cohomology,

$$(\dots) \to \mathrm{H}^i(X, \mathcal{O}(-D)) \to \mathrm{H}^i(X, \mathcal{O}_X) \to \mathrm{H}^i(D, \mathcal{O}_D) \to (\dots)$$

Exercise 2. Let $\iota : Z \to X$ be a closed immersion of schemes, where Z and X are not necessarily noetherian schemes.

(1) Show that the functors $R^j\iota_* : \mathrm{Mod}_{\mathcal{O}_Z} \to \mathrm{Mod}_{\mathcal{O}_X}$ are zero for all $j > 0$.
(2) Conclude that for an $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_Z}$, $\mathrm{H}^i(Z, \mathcal{F}) \cong \mathrm{H}^i(X, \iota_*\mathcal{F})$ for all $i \in \mathbb{N}$.

Exercise 3. *A geometric perspective on the Euler sequence.* Let A be a ring and M a locally free of finite rank A-module.

(1) *Directional derivative.* For a $v \in M$, show that there is a unique A-derivation

$$\frac{\partial}{\partial v}\colon \mathrm{Sym}(M^\vee) \to \mathrm{Sym}(M^\vee)$$

which is equal to the evaluation at v on elements of degree 1. If M is free, if (e_i) and (x_i) denotes a basis and a dual basis respectively, and $v = \sum \lambda_i e_i$, show that

$$\frac{\partial}{\partial v} = \sum_i \lambda_i \frac{\partial}{\partial x_i}.$$

(2) For $\varphi \in M^\vee$, show that $\frac{\partial}{\partial v}$ uniquely extends to an A-derivation

$$\frac{\partial}{\partial v}\colon \mathrm{Sym}(M^\vee)_\varphi \to \mathrm{Sym}(M^\vee)_\varphi.$$

Deduce that $\frac{\partial}{\partial v}$ defines an A-derivation

$$\frac{\partial}{\partial v}\colon \mathrm{Sym}(M^\vee)_{(\varphi)} \to \mathrm{Sym}(M^\vee)(-1)_{(\varphi)}.$$

(3) Denote by $\pi\colon \mathbb{P}(M) \to \operatorname{Spec}(A)$ and $\mathcal{T}^1_{\mathbb{P}(M)|A} = \left(\Omega^1_{\mathbb{P}(M)|A}\right)^\vee$. Deduce that there is a $\mathcal{O}_{\mathbb{P}(M)}$-linear map
$$\frac{\partial}{\partial(-)}\colon \pi^*M \to \mathcal{T}^1_{\mathbb{P}(M)|A}(-1).$$
Hint: $\mathcal{T}^1_{\mathbb{P}(M)|A}(-1) = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}(M)}}(\Omega^1_{\mathbb{P}(M)|A}, \mathcal{O}(-1))$[1]. *Use the universal property of* $\Omega^1_{\mathbb{P}(M)|A}$ *on affines* $D_+(\varphi)$.

(4) *Euler sequence.* Let S be a scheme and $\mathcal{E}$ a locally free sheaf of finite rank on S. Show that there is an exact sequence of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-locally free sheaves
$$0 \to \mathcal{O}(-1) \to \pi^*\mathcal{E} \xrightarrow{\frac{\partial}{\partial(-)}} \mathcal{T}^1_{\mathbb{P}(\mathcal{E})|S}(-1) \to 0$$
where the first arrow is the canonical inclusion $\mathcal{O}(-1) \to \pi^*\mathcal{E}$ and the second is a globalization of the arrow above. *Hint: Use the naturality of the construction to reduce to a case where the base is affine and* $\mathcal{E}$ *is free. We are now working in* $\mathbb{P}^n_A$. *Choose a basis and write the matrices of maps in question on opens* $D_+(x_i)$.

Exercise 4. *Cohomology and affine maps.* Let $X \to Y$ be an affine map of schemes and $\mathcal{F}$ a quasi-coherent sheaf on X.

(1) Show that the natural map $f_* \to Rf_*$ is an isomorphism, meaning that $R^if_* = 0$ if $i > 0$.

(2) Deduce that
$$\mathrm{H}^i(X, \mathcal{F}) \cong \mathrm{H}^i(Y, f_*\mathcal{F}).$$

Exercise 5. *Curves in* $\mathbb{P}^2_k$. Let k be a field. Let $C = V_+(F)$ for $F \in \mathcal{O}_{\mathbb{P}^2_k}(d)(\mathbb{P}^2_k)$ for a $d \geq 1$.

(1) Show that $\mathrm{H}^0(C, \mathcal{O}_C) \cong k$.

(2) Deduce that any C_1 and C_2 of the above form intersect.

(3) Suppose that C does not contain $[0:0:1]$ (this can always be arranged up to an automorphism of $\mathbb{P}^2_k$). Calculate the Čech complex associated to the cover $C \cap D_+(Y) \cup C \cap D_+(X)$ explicitly and deduce that $\mathrm{H}^1(C, \mathcal{O}_C)$ is a k-vector space of dimension $\frac{(d-1)(d-2)}{2}$.

Remark. We say that $\frac{(d-1)(d-2)}{2}$ is the *arithmetic genus* of C. Curves of degree 3 are of arithmetic genus 1. Smooth ones are called *elliptic curves.* Any smooth curve C over an algebraically closed field k with $\mathrm{H}^1(E, \mathcal{O}_E) = 1$ can be realized as a smooth cubic in $\mathbb{P}^2_k$, see for example Harthshorne III,4.6.

Exercise 6. *A Čech cohomology computation.* Let k be a field. Let $U = \mathbb{A}^2_k \setminus 0$. Compute the cohomology of $\mathcal{O}_U$ on U. After showing that $\mathcal{O}_U$ is ample, deduce that Serre vanishing does not hold for U.

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[1] Because in general if $\mathcal{F}$ is finite locally free and $\mathcal{G}$ is a sheaf of $\mathcal{O}$-modules, then $\mathcal{F}^\vee \otimes \mathcal{G} \cong \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$

Exercise 7. *Coherence of derived pushforward:* Let $f : X \to Y$ be a projective morphism between two noetherian schemes. For a coherent $\mathcal{O}_X$-module $\mathcal{F}$, show that $R^i f_* \mathcal{F}$ is coherent for all i.
The exercise will require material taught on 18.12.24.