Dr. Stefano Filipazzi
Dr. Alapan Mukhopadhyay
Léo Navarro Chafloque

EPFL, fall semester 2024
AG II - Schemes and sheaves

Exercises – week 13

Exercise 1. *Separable extensions and differentials.* Let k be a field and l a finite extension. Show that $\Omega^1_{l|k} = 0$ if and only if l is a separable extension.

Exercise 2. *Derivations on an elliptic curve.* Let R be a ring, $P = R[x_1, \ldots, x_n]$ and $P \to A$ a surjection, with kernel I. Recall that by the conormal sequence, if $d\colon I/I^2 \to \bigoplus_{i=1}^n Adx_i$ is given by sending a polynomial to the image of it's derivative then we have an exact sequence

$$I/I^2 \to \bigoplus_{i=1}^{n} Adx_i \to \Omega^1_{A|R} \to 0.$$

We denote by $\mathrm{T}^1_{A|R} = \mathrm{Hom}_A(\Omega^1_{A|R}, A) = \mathrm{Der}_R(A, A)$, the *$A$-module of R-derivations of A.*

(1) Let
$$E = \mathrm{Proj}(\mathbb{C}[X, Y, Z]/(Y^2Z - (X^3 + Z^3))).$$
Denote by x, y the images of $\frac{X}{Z}, \frac{Y}{Z}$ in $A_Z := \mathcal{O}_E(D_+(Z))$ and s, t the images of $\frac{X}{Y}, \frac{Z}{Y}$ in $A_Y := \mathcal{O}_E(D_+(Y))$. Show using the sequence recalled above that (meaning that any derivation is a scalar multiplication of the written generator)
$$\mathrm{T}^1_{A_Z|\mathbb{C}} = A_Z(2y\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y}) \quad \mathrm{T}^1_{A_Y|\mathbb{C}} = A_Y((3t^2 - 1)\frac{\partial}{\partial s} - 3s^2\frac{\partial}{\partial t}).$$

(2) Moreover show that the generators displayed above agree on the intersection $D_+(YZ)$, giving a non-vanishing global section π of $\mathrm{T}_{E|\mathbb{C}} := \mathrm{Hom}_{\mathcal{O}_E}(\Omega^1_{E|\mathbb{C}}, \mathcal{O}_E)$ implying that
$$\mathrm{T}_{E|\mathbb{C}} = \mathcal{O}_{E|\mathbb{C}}\pi.$$

Exercise 3. *Relative* Spec. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_S$-algebra. This means that it is a sheaf $\mathcal{O}_S$-algebras which is quasi-coherent as an $\mathcal{O}_S$-module.

(1) Let $V \subset U \subset S$ two open affines. Show that the diagram

$$\begin{tikzcd} \mathrm{Spec}(\mathcal{A}(V)) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathcal{A}(U)) \arrow[d] \\ V \arrow[r] & U \end{tikzcd}$$

is cartesian.

(2) Let $X = \bigcup U_i$ be an affine cover. Deduce that we can glue the schemes $(\mathrm{Spec}(\mathcal{A}(U_i)))$ to an S-scheme

$$\underline{\mathrm{Spec}}_S(\mathcal{A}) \to S.$$

(3) Show that $\underline{\mathrm{Spec}}_S(\mathcal{A})$ satisfies the following universal property in the category of S-schemes. If $f\colon T \to S$ is an S-scheme then a S-morphism $T \to \underline{\mathrm{Spec}}_S(\mathcal{A})$ is the same as a morphism of $\mathcal{O}_T$-algebras $f^*\mathcal{A} \to \mathcal{O}_T$. Deduce that $\underline{\mathrm{Spec}}_S(\mathcal{A})$ is independent of the affine cover for the construction.

(4) Let $f\colon X \to Y$ be an affine morphism of schemes. Show that there is a natural isomorphism of Y-schemes $X \cong \underline{\mathrm{Spec}}_Y(f_*\mathcal{O}_X)$.

(5) Let $\mathcal{E}$ be a locally free sheaf of finite rank on S. We define

$$\mathbb{V}(\mathcal{E}) = \underline{\mathrm{Spec}}_S(\mathrm{Sym}(\mathcal{E}^\vee))$$

where the $\mathcal{O}_S$-algebra $\mathrm{Sym}(\mathcal{E}^\vee)$ denotes the free $\mathcal{O}_S$-algebra generated by $\mathcal{E}^\vee$.[1] Show that a S-morphism from $f\colon T \to S$ to $\mathbb{V}(\mathcal{E})$ is the same as a global section of $f^*(\mathcal{E})$, *i.e* an element of $f^*(\mathcal{E})(T)$.

(6) Show that there is always a canonical section of $p\colon \mathbb{V}(\mathcal{E}) \to S$ which correspond to $0 \in \mathcal{E}(S)$ which defines a closed subscheme of $\mathbb{V}(\mathcal{E})$ isomorphic to S. We call this closed subscheme the *zero section of* $\mathbb{V}(\mathcal{E})$.

Exercise 4. *Projective bundles.* Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. Let $\mathcal{E}$ be a locally free sheaf of finite rank on S.

(1) Let $V \subset U \subset S$ two open affines. Show that the diagram

$$\begin{tikzcd}
\mathrm{Proj}(\mathcal{A})(V)) \arrow[r] \arrow[d] & \mathrm{Proj}(\mathcal{A})(U)) \arrow[d] \\
V \arrow[r] & U
\end{tikzcd}$$

is cartesian.

(2) Let $X = \bigcup U_i$ be an affine cover. Deduce that we can glue the schemes $(\mathrm{Proj}(\mathcal{A})(U_i)))$ to an S-scheme (*the relative* Proj)

$$\pi\colon \underline{\mathrm{Proj}}(\mathcal{A}) \to S.$$

When $\mathcal{A} = \mathrm{Sym}(\mathcal{E}^\vee)$ we denote $\underline{\mathrm{Proj}}(\mathrm{Sym}(\mathcal{E}^\vee)) = \mathbb{P}(\mathcal{E})$, the *projective bundle associated to* $\mathcal{E}$.

(3) Show that $\mathbb{P}(\mathcal{E})$ satisfies the following universal property in the category of S-schemes. If $f\colon T \to S$ is an S-scheme then a S-morphism $T \to \mathbb{P}(\mathcal{E})$ is the same as a sub-line bundle[2] $\mathcal{L} \subset f^*\mathcal{E}$.
Hint: Show that the line bundles $\mathcal{O}(1)$ on $\mathrm{Proj}(\mathrm{Sym}(\mathcal{E}^\vee)(U))$ glue naturally to a line bundle $\mathcal{O}(1)$ with a surjection

$$\pi^*\mathcal{E}^\vee \to \mathcal{O}(1).$$

[1] It's a gluing of the usual construction in linear algebra.

[2] a subsheaf which is a line bundle, and such that $f^*\mathcal{E}/\mathcal{L}$ is locally free.

The identity correspond therefore to the dual inclusion $\mathcal{O}(-1) \subset \pi^*\mathcal{E}$. Recall that for locally free sheaves of finite rank, pullback and dual naturally commute.

(4) Show that the surjection
$$\mathrm{Sym}(\mathcal{E}^\vee \oplus \mathcal{O}_S) \to \mathrm{Sym}(\mathcal{E}^\vee)$$
induces a closed immersion
$$\mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E} \oplus \mathcal{O})$$
and that the open complement identifies to $\mathbb{V}(\mathcal{E})$, leading to an open-closed decomposition
$$\mathbb{P}(\mathcal{E} \oplus \mathcal{O}) = \mathbb{V}(\mathcal{E}) \sqcup \mathbb{P}(\mathcal{E}).$$
Remark. This generalizes the open closed decomposition $\mathbb{P}_k^{n+1} = \mathbb{A}_k^{n+1} \sqcup \mathbb{P}_k^n$. We can therefore interpret $\mathbb{P}(\mathcal{E} \oplus \mathcal{O})$ as a compactification of $\mathbb{V}(\mathcal{E})$ where we add an ∞-point to each line in $\mathbb{V}(\mathcal{E})$, namely the corresponding point in $\mathbb{P}(\mathcal{E})$.

(5) Show that $\mathcal{O} \subset \mathcal{E} \oplus \mathcal{O}$ defines a section of $\mathbb{P}(\mathcal{E} \oplus \mathcal{O}) \to S$ which leads to an open-closed decomposition
$$\mathbb{V}(\mathcal{O}(1)) \sqcup S = \mathbb{P}(\mathcal{E} \oplus \mathcal{O}).$$
Remark. This generalizes
$$\mathbb{P}_k^{n+1} = \left(\bigcup_{i=0}^{n} D_+(x_i)\right) \sqcup [0 : \ldots 0 : 1].$$

Exercise 5. *Tautological line bundle.* This exercise is a direct follow-up to the preceding one. We call
$$\mathcal{O}(-1) \subset \pi^*\mathcal{E}$$
the *tautological line bundle.* We gather in this exercise various properties of this *universal* line bundle.
Say U is an affine of S, $M = \mathcal{E}(U)$ and $\varphi \in M^\vee$. Let $c \in M \otimes M^\vee$ be the canonical element (corresponding to the identity along the natural isomorphism $M \otimes M^\vee \cong \mathrm{Hom}_A(M, M)$).

(1) Show that $\mathcal{O}(-1)$ can be realized as the sub-line bundle of $\pi^*\mathcal{E}$ generated on $D_+(\varphi)$ by
$$c/\varphi \in \pi^*M(D_+(\varphi)) = \mathrm{Sym}(M^\vee)_{(\varphi)} \otimes M.$$
(2) Let $f\colon T \to S$ an S-scheme. From the previous exercise, deduce that if $T \to \mathbb{P}(\mathcal{E})$ is the map of S-schemes corresponding to an $\mathcal{L} \subset \pi^*\mathcal{E}$, then the following square
$$\begin{tikzcd} \mathbb{V}(\mathcal{L}) \arrow[r] \arrow[d] & \mathbb{V}(\mathcal{O}(-1)) \arrow[d] \\ T \arrow[r] & \mathbb{P}(\mathcal{E}) \end{tikzcd}$$
is Cartesian.

Remark. The above says that $\mathbb{P}(\mathcal{E})$ is the *moduli space of sub-line bundles of* $\mathcal{E}$, and that $\mathcal{O}(-1)$ is the *universal line bundle on the moduli.*

(3) Show that $\mathbb{V}(\mathcal{O}(-1))$ is a closed subscheme of $\mathbb{V}(\pi^*\mathcal{E}) = \mathbb{V}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{E})$. This comes from the surjection $\pi^*\mathcal{E}^\vee \to \mathcal{O}(1)$.

(4) Let $f\colon T \to S$ an S-scheme. Show that a map of S-schemes $T \to \mathbb{V}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{E})$ which corresponds to a pair $(\mathcal{L}, v)$ with $\mathcal{L} \subset f^*\mathcal{E}$ and $v \in f^*\mathcal{E}(T)$ factors through $\mathbb{V}(\mathcal{O}(-1))$ if and only if $v \in \mathcal{L}(T)$.

Remark. In particular if $S = \mathrm{Spec}(k)$ where k is a field, and $\mathcal{E} = k^{n+1}$, the bundle $\mathbb{V}(\mathcal{O}(-1))$ is realized as a closed subscheme of $\mathbb{A}^{n+1}_k \times_k \mathbb{P}^n_k$.

Exercise 6. *Stability properties of (very-)ample sheaves under tensor product.* Let X be a Noetherian scheme. Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on X.

(1) If $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, show that $\mathcal{L} \otimes \mathcal{M}$ is ample.
(2) If $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, deduce that there is a n such that $\mathcal{L}^n \otimes \mathcal{M}$ is ample.
(3) Show that if $\mathcal{L}$ and $\mathcal{M}$ are ample, then $\mathcal{L} \otimes \mathcal{M}$ is ample.

Now suppose that X is an A-scheme where A is a Noetherian ring.

(4) If $\mathcal{L}$ is A-very ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is A-very ample.
(5) If $\mathcal{L}$ is ample, then there is a $n_0 > 0$ such that $\mathcal{L}^n$ is A-very-ample for all $n \geq n_0$.