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AG II - Schemes and sheaves

Exercises – week 10

Exercise 1. *Functoriality of $\mathcal{O}(n)$.* Let R and S be $\mathbb{N}$-graded rings, where R is generated in degree 1, so that $\mathcal{O}(n)$ is a line bundle for each $n \in \mathbb{Z}$. Let $f\colon R \to S$ be an homogeneous map of degree $d \geq 1$, see Exercise 5, week 4. Denote by $g\colon U \to \mathrm{Proj}(R)$ the induced map at Proj from functoriality of Proj. Show then that for $n \geq 0$ we have $g^*\mathcal{O}(n) = \mathcal{O}(nd)_{|U}$.
Hint: check the claim on cocycles.

Exercise 2. *A principal divisor is effective where it has no poles.* Let X be a Noetherian, normal and integral scheme. Recall that for normal Noetherian domain A, the ring A is the intersection of $A_{\mathfrak{p}}$ where $\mathrm{ht}(\mathfrak{p}) = 1$.
Let $f \in K(X)$. Let $U \subset X$ open. Show that if $\mathrm{div}(f)_{|U} \geq 0$ then $f \in \mathcal{O}_X(U)$. If $\mathrm{div}(f)_{|U} = 0$ then $f \in \mathcal{O}(U)^{\times}$.

Exercise 3. *Divisors that are not Cartier.* Let k be a field and $X = V(xy - zw)$ in $\mathbb{A}^4_k$. Note that X is integral and regular in codimension 1.

(1) Show that the closed subsets in X defined by $x = z = 0$ and $x = w = 0$ are prime divisors that are not Cartier. Denote by D_z and D_w these divisors.
(2) Show that $D_z + D_w$ is a Cartier divisor.

Exercise 4. *Exact sequence for class groups.* Let X be an integral separated scheme which is regular in codimension 1. Let Z be a proper closed subset of X and $U = X \setminus Z$.

(1) Show that $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ defined by $\sum n_i D_i \mapsto \sum n_i (D_i \cap U)$ is surjective.
(2) If $\mathrm{codim}(Z, X) \leq 2$, show that that this map is also injective.
(3) If $\mathrm{codim}(Z, X) = 1$ and Z is irreducible, show that there is an exact sequence
$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 1$$
where $\mathbb{Z} \to \mathrm{Cl}(X)$ send 1 to Z.
(4) Let k be a field. Let Z be the zero set of an irreducible homogeneous polynomial of degree d in $\mathbb{P}^n_k$. Deduce that $\mathrm{Cl}(\mathbb{P}^n_k \setminus Z) \cong \mathbb{Z}/d\mathbb{Z}$.

Hint: You may look at chapter II.6 of Hartshorne.

Exercise 5. Let A be a ring and $R_\bullet$ the graded ring $A[x_0, \ldots, x_n]$ with $\deg(x_i) = 1$. Show that the natural map
$$R_m \to \Gamma(\mathrm{Proj}(R), \mathcal{O}(m))$$

is an isomorphism for $m \in \mathbb{Z}$. *Hint: Use the usual cover and the sheaf property.*

Exercise 6. *Support of coherent sheaves.* We define

$$\mathrm{supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}$$

(1) Let A be a ring and M a finitely generated module. Show that $\mathrm{supp}(M)$ is closed.
(2) In the same setup as in item (1), show that $\mathrm{supp}(M) = V(\mathrm{Ann}(M))$, where
$$\mathrm{Ann}(M) = \{f \in A \mid fM = 0\}.$$
(3) Let A be a Noetherian ring, $f \in A$ and M be a finitely generated module. Show $\mathrm{Ann}(M)_f = \mathrm{Ann}(M_f)$.
(4) Let X be a locally Noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. Using the preceding point, define a quasi-coherent sheaf of ideals $\mathrm{Ann}(\mathcal{F})$. Show that $V(\mathrm{Ann}(\mathcal{F})) = \mathrm{supp}(\mathcal{F})$.

Remark. In this case, we then call $V(\mathrm{Ann}(\mathcal{F}))$ with it's natural scheme structure coming from the quasi-coherent sheaf of ideals $\mathrm{Ann}(\mathcal{F})$ the scheme theoretic support of $\mathcal{F}$.

Exercise 7. *Torsion free sheaves.* Let X be an integral scheme with generic point η. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is torsion free if $\mathcal{F}(U)$ is a torsion free $\mathcal{O}(U)$-module for all opens $U \subset X$.

(1) Let $\mathcal{F}$ be any quasi-coherent sheaf. Show that $\mathcal{F}_{tors} \subset \mathcal{F}$, where $s \in \mathcal{F}_{tors}(U)$ if $s \mapsto 0$ along $\mathcal{F} \to \mathcal{F}_\eta$, is a quasi-coherent sheaf and that $\mathcal{F}/\mathcal{F}_{tors}$ is torsion free.
(2) Show that a map between torsion free sheaves is injective if and only if it is injective at a stalk at some point $x \in X$.
(3) Deduce that a map between locally free sheaves of rank 1 is injective or zero.

Exercise 8. *Generic flatness.* Let X be a connected reduced Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X.

(1) Show that there is an non empty open U such that $\mathcal{F}$ is locally free (possibly zero).
   Use Exercise 4.(3), week 9.
(2) Show by Noetherian induction[1] on X that there is a finite partition of X by locally closed subschemes (X_i) with the reduced scheme structure such that $\mathcal{F}$ is locally free when restricted (meaning taking the pullback) to X_i.

Exercise to hand in. *Morphisms and maps between projective spaces.* (Due 2 December, 18:00) Please write your solution in TEX.

[1] see *Hartshorne, II.3.16*

Let k be a field, and $m < n$ two positive integers. Consider the two k-algebra morphisms given by the natural inclusion $\phi\colon k[x_0, \dots, x_m] \hookrightarrow k[x_0, \dots, x_n]$ and the natural quotient $\psi\colon k[x_0, \dots, x_n] \twoheadrightarrow k[x_0, \dots, x_m]$. Let $f\colon \mathbb{A}^{n+1}_k \to \mathbb{A}^{m+1}_k$ and $g\colon \mathbb{A}^{m+1}_k \to \mathbb{A}^{n+1}_k$ denote the corresponding morphisms of affine spaces, and let $\pi\colon \mathbb{P}^n_k \dashrightarrow \mathbb{P}^m_k$ and $\iota\colon \mathbb{P}^m_k \dashrightarrow \mathbb{P}^n_k$ be the corresponding rational maps between projective spaces, obtained by functoriality of Proj, see Exercise 5, week 5.

(1) Assuming that k is algebraically closed so that we can represent closed points with Cartesian coordinates $(a_0, \dots, a_m)$ and $(b_0, \dots, b_n)$, describe the morphisms f and g at the level of coordinates.
(2) Show that ι is a morphism and a closed embedding, and show that π is not everywhere defined. Furthermore, show that the locus where π is not defined is a copy of $\mathbb{P}^{n-m-1}_k$. Lastly, assuming that k is algebraically closed so that we can represent closed points with projective Cartesian coordinates $[a_0 : \ldots : a_m]$ and $[b_0 : \ldots : b_n]$, describe the two maps at the level of coordinates.

In general, π is called *projection from an* $(n - m - 1)$*-plane.* The fibers over closed points of this rational map (i.e., the closure of the fibers of the morphism defined on the domain of π) are copies of $\mathbb{P}^{n-m-1}_k$. For instance, if $n = 2$ and $m = 1$, it is a projection from a the point $[0 : 0 : 1]$.
From the point of view of linear systems (cf. Ch. II.7 in Hartshorne), the rational map π is defined by a proper subspace of $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$, namely by those global sections that vanish along the linear subspace we are projecting from. For instance, in the case $n = 2$ and $m = 1$, the rational map π is defined by considering the sections of $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$ corresponding (cf. Exercise 2) to the lines through the point $[0 : 0 : 1]$.
In the following, $\mathbb{P}^{n-m-1}_k$ will denote the copy of the projective $(n-m-1)$-space along which π is not defined.

(3) Show that $\iota^*\mathcal{O}_{\mathbb{P}^n_k}(1) = \mathcal{O}_{\mathbb{P}^m_k}(1)$. *Hint: you can use Exercise 1.*
(4) Show that $\mathcal{O}_{\mathbb{P}^n}(1)|_{\mathbb{P}^n_k \setminus \mathbb{P}^{n-m-1}_k}$ is isomorphic to $\pi^*\mathcal{O}_{\mathbb{P}^m_k}(1)$. *Hint: you can use Exercise 1.*

In the following, we focus on the case $n = 2$ and $m = 1$, and we further assume that k is algebraically closed. We will denote by $P = [0 : 0 : 1]$ the copy of $\mathbb{P}^{2-1-1}_k$ (i.e., a point) along which π is not defined. We let C_1 be the conic with equation $x_2^2 - x_0x_1 = 0$, which corresponds to the Veronese embedding of $\mathbb{P}^1_k$ in $\mathbb{P}^2_k$ (cf. Exercise 6 in sheet 4)[2]. Then, we denote by C_2 the conic with equation $x_0^2 - x_1x_2 = 0$. Notice that $P \in C_2$ and $P \notin C_1$.

(5) Show that $\mathcal{O}_{\mathbb{P}^2_k}(1)|_{C_1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1_k}(2)$, where we identify $\mathbb{P}^1_k$ with C_1 via the Veronese embedding. *Hint: you can use Exercise 1.*
(6) Show that $\pi|_{C_1}\colon C_1 \to \mathbb{P}^1_k$ is finite of degree 2. *Hint: via isomorphism given by the Veronese embedding, you can identify π_{C_1} with one of the morphisms in Exercise 1 in sheet 7.*
(7) Show that $\pi|_{C_2}\colon C_2 \setminus \{P\} \to \mathbb{P}^1_k$ extends uniquely to an isomorphism $\pi|_{C_2}\colon C_2 \to \mathbb{P}^1_k$. *Hint: Define a map $D_+(x_2) \cap C_2 \to D_+(x_0) \subset \mathbb{P}^1_k$*

[2]More precisely we ware talking about the one induced by Proj by $x_0 \mapsto x_0^2$, $x_1 \mapsto x_1^2$ and $x_2 \mapsto x_0x_1$.

that glues with $\pi|_{C_2} \colon C_2 \setminus \{P\} \to \mathbb{P}^1_k$. *Note that you are forced to send* $\frac{x_1}{x_0} \in K(\mathbb{P}^1_k)$ *to* $\frac{x_1}{x_0} = \frac{x_0}{x_2} \in K(C_2)$ *which ensures unicity.*

(8) Show that $\mathcal{O}_{\mathbb{P}^2_k}(1)|_{C_2}$ is not isomorphic to $(\pi|_{C_2})^*\mathcal{O}_{\mathbb{P}^1_k}(1)$. *Hint: you can use Exercise 1.*

The morphism $\pi|_{C_2}$ is nothing but the *stereographic projection*. Indeed, the fibers of π (i.e., the closure of the fibers of the morphism $\mathbb{P}^2_k \setminus \{P\} \to \mathbb{P}^1_k$) are lines. In the case of C_1, these lines intersect C_1 in 2 (by Bézout's theorem) distinct points, and these points vary as we vary the target point in $\mathbb{P}^1_k$. On the other hand, in the case of C_2, one of the two points is always P. Thus, we get a morphism from $C_2 \setminus \{P\}$ which is an isomorphism with its image, which in turn extends to the whole C_2 (ancient Greeks just settled for a bijection...).

More generally, if we have a regular conic C with a k-rational point P, the projection from P always induces and isomorphism with $\mathbb{P}^1_k$.