Dr. Stefano Filipazzi
Dr. Alapan Mukhopadhyay
Léo Navarro Chafloque

EPFL, fall semester 2024
AG II - Schemes and sheaves

Exercises – week 1

Exercise 1. *Refresh*
The goal of this exercise is to refresh some notions of commutative algebra as well as their interpretation in algebraic geometry. Let k be an algebraically closed field.

(1) Recall (your definition) of $\mathbb{A}^n_k$ and that the polynomnial algebra $k[x_1, \dots, x_n]$ is to be interpreted as *functions* on this space.
(2) A *finite type k-algebra* A is a k-algebra who admits a surjection $f : k[x_1 \dots, x_n] \to A$. The algebra A is to be interpreted as functions on which space ? What is the interpretation of the ideal $\ker(f)$?
(3) Recall what the *localization of a ring on a multiplicative subset* is. Recall that this is an exact functor. Recall the important example of the localization at a prime ideal of a ring.
(4) Let A be a finite type k-algebra. Let $a \in A$. Recall the localization A_a (so with respect to the multiplicative subset $\{a^n\}_{n \geq 0}$). The ring A_a is to be interpreted as functions on which space ? Is $k[x, y]_y$ a finite type k-algebra ?
(5) Let R be a ring R' and R'' some R-algebras. Recall what is the *tensor product* $R' \otimes_R R''$. What is the R-algebra law on this tensor product ? Which is the universal property of this object as an R-algebra ?
(6) Recall that for a ring R, an ideal I of R, multiplicative subset S of R and an R-algebra $\varphi : R \to A$,
$$R/I \otimes_R A \cong A/IA \quad S^{-1}R \otimes_R A \cong \varphi(S)^{-1}A.$$
(7) Let A and B be finite type k-algebras. The k-algebra $A \otimes_k B$ is to be interpreted as the functions on which space ? Now fix surjections $k[x_1, \dots, x_n] \to A$ and $k[x_1, \dots, x_n] \to B$. The k-algebra $A \otimes_{k[x_1,\dots,x_n]} B$ is to be interpreted as the functions on which space ?
(8) Let R be a ring and M an R-module. Recall what is $\mathrm{Ann}(M)$ and show that if $I \leq \mathrm{Ann}(M)$ then M is naturally a R/I-module. Deduce for example that I/I^2 is an A/I module.

Where are we headed? We will introduce the theory of *schemes*. From the course on algebraic curves, you learned how to interpret *finite type k-algebras* as functions on closed subsets of $\mathbb{A}^n_k$, and saw that the study of such spaces was ultimately related to the algebras of their functions. With the theory of schemes, we will now interpret *any commutative ring as functions on some space.* For example, $\mathbb{Z}$ or any ring of integers can be interpreted as functions on some space, and also rings in finite characteristic. This unveils an all new range of geometric objects. One of the strength of the theory of schemes is that it is a general framework which captures not only the geometry of curves over $\mathbb{C}$ but also the geometry of objects that are more arithmetic in nature. The dictionary between algebra and geometry in the setting that you know and was recalled in a small amount in the preceding exercise will extend to the general setting of schemes.

The following exercises are about *sheaves*. Unless specifically mentioned, a sheaf means a *set*-valued sheaf. For a topological space X, $\mathrm{Op}(X)$ denotes the poset of opens of X.

Exercise 2. *$\mathcal{H}om$ sheaf*
Let X be a topological space. Let $\mathcal{F}$ be a presheaf on X and $\mathcal{G}$ be sheaf on X. For any open $U \subset X$, denote by $\mathcal{F}_U$ the presheaf on U defined by $V \mapsto \mathcal{F}(V)$ for any $V \subset U$. Show that the presheaf $\mathcal{H}om : U \mapsto \mathrm{Hom}(\mathcal{F}_U, \mathcal{G}_U)$, where $\mathrm{Hom}(\mathcal{F}_U, \mathcal{G}_U)$ denotes the set of morphisms of (pre)sheaves on U, is a sheaf.

Exercise 3. *Constant sheaves*
Consider the sheafification of the constant presheaf of $\mathbb{Q}$-vector spaces on the real line $\mathbb{R}$ defined by

$$U \in \mathrm{Op}(\mathbb{R}) \mapsto \mathbb{Q}.$$

We denote by this sheafification $\underline{\mathbb{Q}}$. Compute the value of $\underline{\mathbb{Q}}$ on any open subset of the real line. When the dimension of the $\mathbb{Q}$-vector space $\underline{\mathbb{Q}}(U)$ is finite ? In this case, what is this dimension ?

Exercise 4. *Sheaves and sections*

(1) Let X and Y be topological spaces, and $f : Y \to X$ a continuous map. Show that the following

$$\mathcal{F}_f(U) = \{s : U \to Y \text{ continuous} \mid f \circ s = \mathrm{id}_U\}$$

defines a sheaf on X. We call it the sheaf of section of f.

(2) Let $\mathcal{F}$ be a sheaf on a topological space X. Define the topological space

$$|\mathcal{F}| = \bigsqcup_{x \in X} \mathcal{F}_x$$

as a set with the finest topology such that for any $U \subset X$ open and $s \in \mathcal{F}(U)$ the section $x \mapsto s_x$ of the canonical map is continuous. Show that $|\mathcal{F}| \to X$ is a local homeomorphism and that the sheaf of section of this map is isomorphic to $\mathcal{F}$.

Remark. One can promote the construction $\mathcal{F} \mapsto |\mathcal{F}|$ to an equivalence of categories between $\mathrm{Sh}(X)$ and $\mathrm{\acute{E}t}(X)$ the category of local homeomorphisms over X. For more details, see for example *Manifolds, sheaves, and cohomology* by Wedhorn.

Exercise 5. *Sheafification*
Let X be a topological space and $\mathcal{F}$ a presheaf on X. Show that the natural map $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism at stalks.
Find examples of topological spaces X and presheaves $\mathcal{F}$ on X such that

(1) The natural map $\mathcal{F} \to \mathcal{F}^+$ is not injective/resp. not surjective on some non empty open set.
(2) An abelian group valued presheaf with $\mathcal{F} \neq 0$ but $\mathcal{F}^+ = 0$.

Exercise 6. *Some sheaves on the circle*
We are using the notation $\mathcal{F}_f$ from exercise 4.

(1) Consider the map $e : [0, \frac{3}{2}] \to S^1$ defined by $t \mapsto \exp(2\pi i t)$. Compute all stalks of $\mathcal{F}_e$.
(2) Let $\mathcal{O}$ be the presheaf on S^1 defined for $U \in \mathrm{Op}(S^1)$ by
$$\mathcal{O}(U) = \{U \to \mathbb{R} \text{ continuous}\}$$
Show that $\mathcal{O}$ is a sheaf. Note that $\mathcal{O}$ is a sheaf of $\mathbb{R}$-algebras by acting pointwise. Show that for every $z \in S^1$, $\mathcal{O}_z$ is a local $\mathbb{R}$-algebra with residue field $\mathbb{R}$.

Consider now the quotient M of $[0,1] \times \mathbb{R}$ by identifying $(0,t)$ with $(1,-t)$. Consider the map $\pi\colon M \to S^1$ defined by $\pi([x,t]) = \exp(2\pi i x)$. We also take the notation $\mathcal{F}_\pi = \mathcal{L}$.

(3) Show that for every $U \in \mathrm{Op}(S^1)$, $\mathcal{L}(U)$ is an $\mathcal{O}(U)$-module by $\mathcal{O}(U)$ acting on the second component.
(4) Show that for every open set $U \subset S^1$ with at least one point missing there is an isomorphism of sheaves $\mathcal{O}_U \cong \mathcal{L}_U$ which respects the module structure on evaluation on each open subset.
(5) Show that for every $s \in \mathcal{L}(S^1)$ there exist a $z \in S^1$ such that $s(z) = [z, 0]$.
(6) Deduce that there is *no* isomorphism $\mathcal{O} \cong \mathcal{L}$ of sheaves respecting the module structure on each open subset.

Exercise 7. *Skyscraper sheaves*
For any set S and $x \in \mathbb{R}$ show that the following defines a sheaf, for $U \in \mathrm{Op}(\mathbb{R})$
$$x_*S(U) = \begin{cases} S \text{ if } x \in U \\ \mathbb{1} \text{ if } x \notin U, \end{cases}$$
where $\mathbb{1}$ is the set with one element. We call this sheaf the *skyscraper sheaf of S at x*. Compute every stalk of x_*S. Understand and draw the topological space $|x_*S|$ (see exercise 4). Do you understand the name *skyscraper* now ?