# Algebraic Geometry II (MATH-510) — Final exam

22 January 2024, 9 h 15 – 12 h 15

**Name : Empty2**

**SCIPER : 000000**

**Signature :** ______________________________

Number
15

Paper & pen: This booklet contains 6 exercises, on 36 pages, for a total of 100 points. Please use the space with the square grid for your answers. **Do not** write on the margins. Write all your solutions under the corresponding exercise, except if you run out of space at a given exercise. In that case, continue with your solution at the empty space left after your solution for another exercise. In this case, mark clearly where the continuation of your solution is. If even this way the booklet is not enough, then ask for additional papers from the proctors. Write your name and the exercise number clearly on the top right corner of the additional paper. At the end of your exam put the additional papers into the exam booklet under the supervision of a proctor, and sign on to the number of additional papers on the proctor's form. We provide scratch paper. You are not allowed to use your own scratch paper. Please write with a pen, NOT with a pencil.

Duration of the exam: It is not allowed to read the inside of the booklet before the exam starts. The length of the exam is 180 minutes. If you did not leave until the final 20 minutes, then please stay seated until the end of the exam, even if you finish your exam during these 20 minutes. The exams are collected by the proctors at the end of the exam, during which please remain seated.

CAMIPRO & coats: Please prepare your CAMIPRO card on your table. Your bag and coat should be placed close to the walls of the room, NOT in the vicinity of your seat.

Cheat-sheet & results of the course: During the exam, you can use a two-sided cheat-sheet of size A4. Furthermore, you can consult the two copies of Hartshorne at the proctors' desk. No other resource is allowed during the exam. In particular, electronic devices are not allowed.

In your solutions, you can use all the material learned during the lectures and the exercise sessions, including the lecture notes, Hartshorne, and the solutions of the exercise sheets. However, please state always what you are using. If you are using a not very frequently used statement, please give a precise reference, preferably with numbers.

Separate points can be solved separately: You get maximum credit for solving any point of an exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

Assumptions: All rings are commutative and with identity. A variety $X$ over a field $k$ is an integral scheme whose structure morphisms $X \to \mathrm{Spec}(k)$ is separated and of finite type.

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|---|
| Points: | 20 | 15 | 20 | 15 | 15 | 15 | 100 |
| Score: | | | | | | | |

**Exercice 1** [ **20 pts** ]

*Recall that you get maximum credit for solving any point of the exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.*

Let $k$ be an algebraically closed field. Consider the affine scheme corresponding to the $k$-algebra $k[x, y]/(y^2 - x^3 + x)$. Let $\phi\colon k[x] \to k[x, y]/(y^2 - x^3 + x)$ denote the morphism obtained by composing the natural inclusion $k[x] \hookrightarrow k[x, y]$ with the surjection $k[x, y] \twoheadrightarrow k[x, y]/(y^2 - x^3 + x)$. Also, let $\psi\colon k[y] \to k[x, y]/(y^2 - x^3 + x)$ denote the morphism obtained in the analogous way. In the following, let $f\colon \operatorname{Spec}(k[x, y]/(y^2 - x^3 + x)) \to \operatorname{Spec}(k[x])$ and $g\colon \operatorname{Spec}(k[x, y]/(y^2 - x^3 + x)) \to \operatorname{Spec}(k[y])$ denote the corresponding morphisms of schemes.

**Warning:** In the following, some of the answers may depend on $\operatorname{char}(k)$. Watch out!

(1) Show that $f$ is a finite morphism.

In the following, you can freely assume that $f$ and $g$ are finite morphisms.

(2) Compute the following fibers of $f$: over $(x)$, the generic fiber, and the geometric generic fiber. For each of these, also determine the cardinality of each of the above fibers as a set. **Note:** if you need to claim that some polynomial is irreducible, you do not need to prove it (so long as your claim is correct!).

(3) Compute the fiber of $g$ over the point $(y)$, determine the dimension as $k$-vector space of the corresponding algebra, and determine the cardinality of the fiber as a set.

**Exercice 2** [ **15 pts** ]

*Recall that you get maximum credit for solving any point of the exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.*

Let $k$ be an algebraically closed field. In $\mathbb{A}^2_k$, consider the planar curves $C$ and $D$ determined by the ideals $(y^2 - x^3)$ and $(y^2 - x^3 - x^2)$, respectively. Recall that the scheme theoretic intersection $C \cap D$ is given by the sum ideal $(y^2 - x^3, y^2 - x^3 - x^2)$.

(1) Determine the dimension of $k[x, y]/(y^2 - x^3, y^2 - x^3 - x^2)$ as $k$-vector space.

(2) Determine the radical $\sqrt{(y^2 - x^3, y^2 - x^3 - x^2)}$. Set theoretically, how many points are in the intersection $C \cap D$?

(3) Are $C$ and $D$ regular? If yes, prove it. If not, list all of their singular points.

**Exercice 3** [ **20 pts** ]

*Recall that you get maximum credit for solving any point of the exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.*

Let $X$ be a locally Noetherian scheme and $x \in X$. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on $X$, and $\varphi\colon \mathcal{F} \to \mathcal{G}$ a morphism of $\mathcal{O}_X$-modules.

*You can freely use the following lemma.*

**Lemma.** Let $(R, \mathfrak{m})$ be a local ring and $M$ a finitely generated $R$-module. If $\mathfrak{m}M = M$, then $M = 0$.

(1) If $\mathcal{F}_x = 0$, show that there is an open neighborhood $U$ of $x$ such that $\mathcal{F}_U = 0$.

(2) Show that if $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is surjective (resp. injective), then there is an open neighborhood $U$ of $x$ such that $\varphi_U$ is is surjective (resp. injective).

(3) Define $\mathcal{F}(x) = \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$ and $\mathcal{G}(x) = \mathcal{G}_x/\mathfrak{m}_x\mathcal{G}_x$. Denote by $\varphi(x)$ the induced map. Show using the lemma above, which you can use without proof, that if $\varphi(x)$ is surjective, then there is an open neighborhood $U$ of $x$ such that $\varphi_U$ is surjective.

(4) Let $Y \to X$ and $Z \to X$ be two finite schemes over $X$. Show that a morphism of $X$-schemes $f\colon Y \to Z$ is a closed immersion if and only for each point $x \in X$ the pullback map
$$f(x)\colon Y \times_X \operatorname{Spec}(k(x)) \to Z \times_X \operatorname{Spec}(k(x))$$
is a closed immersion.

**Exercice 4** [ **15 pts** ]

*Recall that you get maximum credit for solving any point of the exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.*

Parts (1) and (2) of this exercise are independent of one another.

(1) Let $A$ be a ring. Let $n \geq 1$ be a natural number, and recall the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n_A}(-1) \to \mathcal{O}_{\mathbb{P}^n_A}^{\oplus n+1} \to \Omega^\vee_{\mathbb{P}^n_A/A} \otimes \mathcal{O}_{\mathbb{P}^n_A}(-1) \to 0.$$

Show that

$$H^0(\mathbb{P}^n_A, \Omega^\vee_{\mathbb{P}^n_A/A} \otimes \mathcal{O}_{\mathbb{P}^n_A}(-1)) \cong A^{\oplus n+1}$$

and that

$$H^i(\mathbb{P}^n_A, \Omega^\vee_{\mathbb{P}^n_A/A} \otimes \mathcal{O}_{\mathbb{P}^n_A}(-1)) = 0$$

for $i \geq 1$.

(2) Let $k$ be a field. Consider $\mathbb{P}^5_k = \operatorname{Proj}(k[x_0, \dots, x_5])$. We consider the closed subscheme

$$X = V_+(x_0^2 + x_1 x_2).$$

You can freely use that $X$ is a Cartier divisor in $\mathbb{P}^5_k$ with ideal sheaf isomorphic to $\mathcal{O}_{\mathbb{P}^5_k}(-2)$.

(a) Show that

$$H^i(X, \mathcal{O}_X) = 0$$

if $i > 0$ and that $H^0(X, \mathcal{O}_X) = k$.

(b) Show that for $1 \leq j \leq 3$ we have

$$H^i(X, \mathcal{O}_X(-j)) = 0$$

for all $i \geq 0$.

**Exercice 5** [ **15 pts** ]

*Recall that you get maximum credit for solving any point of the exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.*

Fix a field $k$. For two positive integers $n, m$, by $\mathbb{P}^n_k$ and $\mathbb{P}^m_k$ we denote $\mathrm{Proj}(k[x_0, x_1, \ldots, x_n])$ and $\mathrm{Proj}(k[y_0, y_1, \ldots, y_m])$ respectively. Let $p_1$ and $p_2$ denote the projections of $\mathbb{P}^n_k \times_k \mathbb{P}^m_k$ to $\mathbb{P}^n_k$ and $\mathbb{P}^m_k$ respectively.

(1) Show that $p_1^*\mathcal{O}_{\mathbb{P}^n_k}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^m_k}(1)$ is $k$-very ample, where the tensor product is taken over the structure sheaf of $\mathbb{P}^n_k \times_k \mathbb{P}^m_k$.

(2) Let $X$ and $Y$ be two projective varieties over $k$, with closed immersions $\iota_X : X \to \mathbb{P}^n_k$ and $\iota_Y : Y \to \mathbb{P}^m_k$. Show that $\iota_X \times \iota_Y : X \times_k Y \to \mathbb{P}^n_k \times_k \mathbb{P}^m_k$ is a closed immersion.

(3) Let $X$ and $Y$ be two projective varieties over $k$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be ample invertible $\mathcal{O}_X$ and $\mathcal{O}_Y$ modules respectively. Denote the projections of $X \times_k Y$ to $X$ and $Y$ by $\pi_1$ and $\pi_2$ respectively. Prove that the invertible $\mathcal{O}_{X\times_k Y}$ module $\pi_1^*\mathcal{L}_1 \otimes \pi_2^*\mathcal{L}_2$ is ample, where the tensor product is taken over the structure sheaf of $X \times_k Y$.

**Exercice 6** [ **15 pts** ]

*Recall that you get maximum credit for solving any point of the exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.*

Let $k$ be an algebraically closed field. Set $\mathbb{P}^1_k = \mathrm{Proj}(k[x,y])$. Recall that $\Omega_{\mathbb{P}^1_k/k}$ is an invertible $\mathcal{O}_{\mathbb{P}^1_k}$-module.

(1) Consider $d(x/y) \in \Omega_{\mathbb{P}^1_k/k}(D_+(y))$, where $d\colon \mathcal{O}_{\mathbb{P}^1_k} \to \Omega_{\mathbb{P}^1_k/k}$ is the universal derivation. Compute the divisor on $\mathbb{P}^1_k$ associated to $d(x/y) \in \Omega_{\mathbb{P}^1_k/k}(D_+(y))$.

**HINT:** Working with the affine charts on $\mathbb{P}^1_k$ might be helpful.

(2) Given a divisor $D = \sum_{i=1}^{m} n_i P_i$, where $P_i$'s are closed points on $\mathbb{P}^1_k$, find the integer $j$ such that $\mathcal{O}_{\mathbb{P}^1_k}(D) \cong \mathcal{O}_{\mathbb{P}^1_k}(j)$. Justify your answer.

(3) We know that $\mathrm{Pic}(\mathbb{P}^1_k) \cong \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}^1_k}(1)$. Based on your answer in part (1), determine the integer $i$ such that $\Omega_{\mathbb{P}^1_k/k} \cong \mathcal{O}_{\mathbb{P}^1_k}(i)$ as $\mathcal{O}_{\mathbb{P}^1_k}$-modules.