

Modern algebraic geometry (MATH-510) — Final exam

23 January 2023, 8 h 15 – 11 h 15

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Nom :

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Paper & pen: This booklet contains 5 exercises, on 6 pages, for a total of 100 points. Please use the space with the square grid for your answers. **Do not** write on the margins. Write all your solutions under the corresponding exercise, except if you run out of space at a given exercise. In that case, continue with your solution at the empty space left after your solution for another exercise. In this case, mark clearly where the continuation of your solution is. If even this way the booklet is not enough, then ask for additional papers from the proctors. Write your name and the exercise number clearly on the top right corner of the additional paper. At the end of your exam put the additional papers into the exam booklet under the supervision of a proctor, and sign on to the number of additional papers on the proctor's form. We provide scratch paper. You are not allowed to use your own scratch paper. Please write with a pen, NOT with a pencil.

Duration of the exam: It is not allowed to read the inside of the booklet before the exam starts. The length of the exam is 180 minutes. If you did not leave until the final 20 minutes, then please stay seated until the end of the exam, even if you finish your exam during these 20 minutes. The exams are collected by the proctors at the end of the exam, during which please remain seated.

CAMIPRO & coats: Please prepare your CAMIPRO card on your table. Your bag and coat should be placed close to the walls of the room, NOT in the vicinity of your seat.

Open book exam & results of the course: This is an open book exam. That is, you can have any printed or handwritten help with you, but you cannot use electronic devices.

In your solutions, you can use all the material learned during the lectures and the exercise sessions, including the lecture notes, Hartshorne, and the solutions of the exercise sheets. However, please state always what you are using. If you are using a not very frequently used statement, please give a precise reference, preferably with numbers.

Separate points can be solved separately: You get maximum credit for solving any point of an exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

Assumptions: all rings are commutative and with identity.

Question:	1	2	3	4	5	Total
Points:	15	10	20	25	30	100
Score:						

Exercise 1 [15 pts]

Let k be a field and consider the ideal $I = (x^2yz, xy^2z)$ in the ring $R = k[x, y, z]$.

- (1) Find a minimal primary decomposition of I .
- (2) Compute the dimension of $X = \operatorname{Spec}(R/I)$.
- (3) Assume that k is an algebraically closed field. At which closed points $p \in X$ is the stalk $\mathcal{O}_{X,p}$ reduced?

Exercise 2 [10 pts]

Let $A := \mathbb{Z}[x]/(2x^2 + 1)$. Consider $X := \operatorname{Spec} A$ endowed with the canonical morphism $\pi: X \rightarrow \operatorname{Spec}(\mathbb{Z})$.

- (1) Is the morphism π finite?
- (2) Find the largest open set U of $\operatorname{Spec}(\mathbb{Z})$ such that $\pi|_{\pi^{-1}(U)}: \pi^{-1}(U) \rightarrow U$ is a finite morphism.

Exercise 3 [20 pts]

Recall that you get maximum credit for solving any point of the exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

In this exercise we fix X to be a scheme.

Recall: We write $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ to mean the $\mathcal{O}_X$ -module of $\mathcal{O}_X$ -module homomorphisms as defined in Hartshorne, Definitions Page 109.

For a $\mathcal{O}_X$ -module $\mathcal{F}$, we define the dual $\mathcal{O}_X$ -module by $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

- (1) If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$ -module, show that $\mathcal{F}^\vee$ is also quasi-coherent. (Hint: you can reduce to the case where $X = \operatorname{Spec}(A)$, where A is a ring, and $\mathcal{F} = \widetilde{M}$ where M is a A -module)
- (2) Consider $\mathbb{A}_k^2 = \operatorname{Spec} k[x, y]$, where k is a field. Let $\mathcal{I}_0$ be the $\mathcal{O}_{\mathbb{A}_k^2}$ -module associated to the maximal ideal $(x, y) \subset k[x, y]$. Show that $\mathcal{I}_0^\vee$ is a free sheaf of rank 1 (equivalently $\mathcal{I}_0^\vee \simeq \mathcal{O}_{\mathbb{A}_k^2}$).
- (3) Show that if $\mathcal{E}$ is a locally free $\mathcal{O}_X$ -module, then $\mathcal{E}^\vee$ is also locally free.
- (4) If $\mathcal{E}^\vee$ is locally free, is $\mathcal{E}$ locally free? Prove it, or disprove it with an example.

Exercise 4 [25 pts]

Recall that you get maximum credit for solving any point of the exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

Let $p > 0$ be a prime number and let $k = \mathbb{F}_p(t) = \text{Frac } \mathbb{F}_p[t]$. Let $A := k[x, y]/(y^p - x^{p+1} - t)$ and $X := \text{Spec } A$.

- (1) Show that the ideal $\mathfrak{m} = (x)$ is a maximal ideal in A .
- (2) Show that the sheaf of Kähler differentials $\Omega_{X/k}^1$ is not a locally free $\mathcal{O}_X$ -module.
- (3) Consider the base change $Y := X \times_{\text{Spec}(k)} \text{Spec}(A/\mathfrak{m})$. Prove that Y is not regular.

Exercise 5 [30 pts]

Recall that you get maximum credit for solving any point of the exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

Let k be a field. Let $A := k[x, y, z, w]$ and consider the ideal $I := (x, y, z) \subset A$. Consider the graded A -module $S := \bigoplus_{n \geq 0} I^n$, where I^n is the n -th power of the ideal.

Recall from lectures that

- (a) there is a canonically defined morphism $\pi: \operatorname{Proj} S \rightarrow \operatorname{Spec}(A) = \mathbb{A}_k^4$;
- (b) by Corollary II.5.16 of Hartshorne, a closed subscheme Y of $\mathbb{P}_A^r = \operatorname{Proj} A[y_0, \dots, y_r]$ is defined by a homogeneous ideal of $A[y_0, \dots, y_r]$.

Prove the following:

- (1) Show there is a natural closed immersion $\varphi: \operatorname{Proj} S \hookrightarrow \mathbb{P}_A^2 = \operatorname{Proj} A[X, Y, Z]$ of A -schemes and compute the homogeneous ideal in $A[X, Y, Z]$ defining $\operatorname{Proj} S$.
- (2) Suppose from now on that k is an algebraically closed field and compute the fibre of $\pi: \operatorname{Proj} S \rightarrow \operatorname{Spec} A$ over any closed point p of $\operatorname{Spec}(A)$. (Hint: distinguish whether p belongs to $V(I)$ or not).
- (3) Prove that π is a birational morphism over k .